Math 582: Ordinal Arithmetic

Associative Laws

$$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma) \qquad (\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)$$

Not Commutative Laws

$$1+\omega=\omega<\omega+1 \qquad 2\cdot\omega=\omega<\omega\cdot 2$$

Left Distributive Law: $\alpha\cdot(\beta+\gamma)=\alpha\cdot\beta+\alpha\cdot\gamma$

Not Right Distributive Law: $(1+1)\cdot\omega=\omega<\omega+\omega=1\cdot\omega+1\cdot\omega$

Zero and One Laws

$$\begin{aligned} \alpha+0=0+\alpha&=\alpha \qquad \alpha\cdot 0=0\cdot\alpha=0\\ S(\alpha)&=\alpha+1 \qquad \alpha\cdot 1=1\cdot\alpha=\alpha\\ \alpha^0&=1 \qquad \alpha^1=\alpha \qquad 1^\alpha=1\\ \alpha>0\ \to\ 0^\alpha&=0 \end{aligned}$$

Left Cancellation

$$\begin{aligned} \alpha+\beta=\alpha+\gamma &\quad\to\quad \beta=\gamma\\ \alpha\cdot\beta=\alpha\cdot\gamma &\quad\to\quad \beta=\gamma \end{aligned}$$

Not Right Cancelation

$$1+\omega=2+\omega \qquad 1\cdot\omega=2\cdot\omega$$

Subtraction

$$\alpha\le\beta \quad\to\quad \exists!\gamma\,\alpha+\gamma=\beta$$

Division

$$\alpha>0 \quad\to\quad \exists!\gamma\delta\,\big(\beta=\alpha\cdot\gamma+\delta\ \wedge\ \delta<\alpha\big)$$

Exponentiation

$$\alpha^{\beta+\gamma}=\alpha^\beta\cdot\alpha^\gamma \qquad \alpha^{\beta\cdot\gamma}=(\alpha^\beta)^\gamma$$

Logarithms

$$\alpha>1\ \wedge\ \beta>0 \quad\to\quad \exists!\gamma\delta\xi\,\big(\beta=\alpha^\delta\cdot\xi+\gamma\ \wedge\ \xi<\alpha\ \wedge\ \gamma<\alpha^\delta\ \wedge\ \xi>0\big)$$

For finite ordinals, $\delta=\lfloor\log_\alpha\beta\rfloor$. For example, $873=10^2\cdot 8+73\ \wedge\ 8<10\ \wedge\ 73<10^2$.

Order

$$\begin{aligned} \beta<\gamma &\quad\to\quad \alpha+\beta<\alpha+\gamma\ \wedge\ \beta+\alpha\le\gamma+\alpha\\ \beta<\gamma\ \wedge\ \alpha>0 &\quad\to\quad \alpha\cdot\beta<\alpha\cdot\gamma\ \wedge\ \beta\cdot\alpha\le\gamma\cdot\alpha\\ \beta<\gamma\ \wedge\ \alpha>1 &\quad\to\quad \alpha^\beta<\alpha^\gamma\ \wedge\ \beta^\alpha\le\gamma^\alpha \end{aligned}$$

The $\le$ are required: $2<3$ but $2+\omega=3+\omega$, $2\cdot\omega=3\cdot\omega$, $2^\omega=3^\omega$.