Math 582: Axioms of Set Theory

Axioms stated with free variables are understood to be universally quantified.

Axiom 0. Set Existence

$$\exists x(x = x)$$

Axiom 1. Extensionality

$$\forall z(z \in x \leftrightarrow z \in y) \;\rightarrow\; x = y$$

Axiom 2. Foundation

$$\exists y(y \in x) \quad \rightarrow \quad \exists y\big(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)\big)$$

Axiom 3. Set Comprehension Scheme For each formula φ, without y free,

$$\exists y \forall x\big(x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x)\big)$$

Axiom 4. Pairing

$$\exists z(x \in z \wedge y \in z)$$

Axiom 5. Union

$$\exists A\, \forall x\big(\,\exists Y \in \mathcal{F}(x \in Y) \quad \rightarrow \quad x \in A\big)$$

Axiom 6. Replacement Scheme For each formula φ, without B free,

$$\forall x \in A\, \exists! y\, \varphi(x, y) \quad \rightarrow \quad \exists B\, \forall x \in A\, \exists y \in B\, \varphi(x, y)$$

The rest of the axioms are easier to state using some defined notions. On the basis of Axioms 1, 3, 4, 5 we can define $\subseteq$ (subset), $\emptyset$ (emptyset), S (ordinal successor function), $\cap$ (intersection), and SING(x) (x is a singleton) by:

$$\begin{aligned}
x \subseteq y &\iff \forall z(z \in x \rightarrow z \in y) \\
x = \emptyset &\iff \forall z(z \notin x) \\
y = S(x) &\iff \forall z(z \in y \;\leftrightarrow z \in x \vee z = x) \\
w = x \cap y &\iff \forall z(z \in w \;\leftrightarrow z \in x \wedge z \in y) \\
\mathrm{SING}(x) &\iff \exists y \in x\, \forall z \in x(z = y)
\end{aligned}$$

Axiom 7. Infinity

$$\exists x\big(\emptyset \in x \wedge \forall y \in x(S(y) \in x)\big)$$

Axiom 8. Power Set

$$\exists y \forall z(z \subseteq x \;\rightarrow\; z \in y)$$

Axiom 9. Choice

$$\emptyset \notin A \wedge \forall x, y \in A(x \neq y \ \rightarrow \ x \cap y = \emptyset) \quad \rightarrow \quad \exists C \, \forall x \in A(\mathrm{SING}(C \cap x))$$

We will be considering various collections of axioms ("Z" stands for Zermelo and "F" stands for Frankel):

- ZFC = Axioms 1-9
- ZF = Axioms 1-8
- ZC and Z are ZFC and ZF, respectively, with Axiom 6 (Replacement) removed.
- $Z^{-}, ZF^{-}, ZC^{-}, ZFC^{-}$ are Z, ZF, ZC, ZFC, respectively, with Axiom 2 (Foundation) removed.

Most of elementary mathematics takes place within ZC^{-}. The standard set of axioms for set theory is ZFC, Zermelo-Frankel Set Theory with Choice.