Math 582
Introduction to Set Theory

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January 28, 2009

Set Existence

Axiom 0: Set Existence:

$$\exists x(x = x)$$

- This axiom is not technically needed since $\exists x(x = x)$ is a logical truth. (Logic presupposes that the universe of objects is not empty. There are is a development of logic, called free logic, which allow empty universes.)
- The set given by the axiom may have members, but its only use will be to allow us to show there is an emptyset (using Comprehension.)
- Hrbacek/Jech take the existence of the empty set as an axiom, and call our axiom the Weak Set Existence Axiom.

Extensionality

Axiom 1: Extensionality:

$$\forall x, y\, \big(\forall z(z \in x \leftrightarrow z \in y) \;\rightarrow\; x = y\big)$$

The main use of Extensionality is to guarantee that the sets we introduce through Comprehension are unique (see Lemma 3.4 of H/J, p. 9):

Lemma. Let φ be a formula. If A, B are sets such that

$$\forall x\big(x \in A \;\leftrightarrow\; \varphi(x)\big)$$
$$\forall x\big(x \in B \;\leftrightarrow\; \varphi(x)\big)$$

then $A = B$.

Comprehension

Axiom 3: Comprehension: For each formula φ, without y free,

$$\exists y \forall x\big(x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x)\big)$$

(The axiom is understood to be universally quantified, so there are no free variables. In particular, $\forall z$ is one of the quantifiers.)

- The only restriction on φ in the comprehension principle, is that the set defined by the axiom, y, does not occur free in φ. This prevents circular (and contradictory) instances:

$$\exists y \forall x\big(x \in y \quad \leftrightarrow \quad x \in z \wedge x \notin y\big)$$

No such set y can exists, for any nonempty set z.

How we use Comprehension

We will introduce new sets in a four step process:

1. Use Axioms 0,4,5,6,7,8 to get a set B which is *big enough* to collect all sets which satisfy some property φ.
2. Use Comprehension to get a set A which contains only sets x which satisfy $\varphi(x)$:
$$\exists A \forall x \big(x \in A \quad \leftrightarrow \quad x \in B \wedge \varphi(x) \big)$$
3. Use Extensionality to show A is the unique set which collects the sets satisfying $\varphi(x)$.
4. Introduce a name for A: $\{x \mid \varphi(x)\}$.

Example: The empty set

1. By Set Existence there is a set B.
2. By Comprehension there is a set A satisfying
$$\forall x \big(x \in A \quad \leftrightarrow \quad x \in B \wedge x \neq x \big)$$
So, A is an empty set: $\forall x (x \notin A)$. Thus, it is true that
$$\forall x \big(x \in A \quad \leftrightarrow \quad x \neq x \big)$$
(i.e. the extra condition $x \in B$ is unnecessary)
3. Suppose C is also an emptyset, $\forall x (x \notin C)$. Then
$$\forall x \big(x \in C \quad \leftrightarrow \quad x \neq x \big)$$
So, $A = C$ by Extensionality.
4. Let $A = \{x \mid x \neq x\}$ We also use the name for this set.

Set Abstraction

For any formula φ,

- If we can prove there is a set A such that

  $$\forall x\big(x \in A \quad \leftrightarrow \quad \varphi(x)\big)$$

  then A is the unique set which collects all and only x satisfying $\varphi(x)$ (by Extensionality), and we will denote A by

  $$\{x \,|\, \varphi(x)\}$$

- $\{x \in z \,|\, \varphi(x)\}$ abbreviates $\{x \,|\, x \in z \wedge \varphi(x)\}$.

Proper Classes

For any formula φ,

- If we can prove there is no set A such that

  $$\forall x\big(x \in A \quad \leftrightarrow \quad \varphi(x)\big)$$

  then we will call the collection

  $$A = \{x \,|\, \varphi(x)\}$$

  a proper class. Formally, A does not exist (that is, there is no set in our universe of sets which collects all and only sets x satisfying ϕ.)

- Informally, it is still useful to talk about proper classes. For example, we would like to talk about the universe of sets, $V = \{x \,|\, x = x\}$, but there is no such universal set. There are versions of set theory which allow us to introduce proper classes formally into the language of set theory (von Neuman-Gödel-Bernays class theory and Kelley-Morse class theory.) These theories place severe limitations on proper classes – a class is never a member of a set or another class, for example.

Proper Classes

Theorem. There is no universal set: $\forall x \exists z (z \notin x)$.

Proof.

Let x be any set and let $R = \{z \in x \mid z \notin z\}$ (by Comprehension.) Then

$$(*) \quad R \in R \leftrightarrow R \in x \wedge R \notin R$$

Suppose $R \in R$. Then $R \in x \wedge R \notin R$ by (*), which is impossible.
So, $R \notin R$. Thus, either $R \notin x$ or $R \in R$ (by (*).) Since it is impossible that $R \in R$, we must have $R \notin x$. □

Note that the R in this theorem is similar to the Russell set, but constrained to the set x; the conclusion is that $R \notin x$. In Naive set theory we could take x to be the universal (naive) set V, so $R \notin V$ – which is impossible since R is a set!!

Conditions on φ in Comprehension

Axiom 3: Comprehension: For each formula φ, without y free,

$$\exists y \forall x \big(x \in y \quad \leftrightarrow \quad x \in z \wedge \varphi(x)\big)$$

(The axiom is understood to be universally quantified, so there are no free variables. In particular, $\forall z$ is one of the quantifiers, as well as quantifiers for each parameter in φ.)

- There may be other variables besides x free in φ (including the set z). For example, for any sets u, z

$$\begin{aligned} z \cap u &= \{x \mid x \in z \wedge x \in u\} && \varphi(x) \text{ is } x \in u \\ z - u &= \{x \mid x \in z \wedge x \notin u\} && \varphi(x) \text{ is } x \notin u \\ z^* &= \{x \in z \mid x \cap z \neq \emptyset\} && \varphi(x) \text{ is } x \cap z \neq \emptyset \end{aligned}$$

(For example, if $z = \{\emptyset, \{\emptyset\}\}$ then $z^* = \{\{\emptyset\}\}$, since $\{\emptyset\} \cap z = \{\emptyset\}$ and $\emptyset \cap z = \emptyset$.)

Universe of sets

☞ The only sets we can prove exists from just Axioms 0, 1, 3 is $\emptyset$.

See this week's homework assignment for a model of Axioms 0, 1, 2, 3 whose only set is the emptyset.

Pairing

Axiom 4: Pairing:

$$\forall x, y \exists z (x \in z \wedge y \in z)$$

- The set z given by Pairing may contain other elements besides x, y. The axiom only guarantees that there is a set z *big enough* to contain both x and y. We need Comprehension to form $\{x, y\}$.
- This is called Weak Pairing Axiom in H/J (where their axiom gives the pair $\{x, y\}$ directly without using Comprehension.)

Definitions

Definition. Let x, y be sets. Then the following are sets

- $\{x, y\} = \{w \mid w = x \vee w = y\}$.
- $\{x\} = \{x, x\}$.
- $(x, y) = \langle x, y \rangle = \{\{x\}, \{x, y\}\}$. (Our "official" definition of ordered pair.)

The key property about ordered pair is that x, y are uniquely determined:

Theorem

For sets x, y, x', y'

$$\langle x, y \rangle = \langle x', y' \rangle \ \rightarrow \ x = x' \wedge y = y'$$

Axiom 4: Pairing

A word on ordered pairs

There are many other definitions for ordered pair which satisfies the key property :

$$\langle x, y \rangle = \langle x', y' \rangle \ \rightarrow \ x = x' \wedge y = y'$$

For example, the following works just as well:

$$\langle\langle x, y \rangle\rangle = \{\{\emptyset, x\}, \{\{\emptyset\}, y\}\}$$

It usually does not matter what definition is used, provided x and y are uniquely determined from the ordered pair. It is conventional to use the notation (x, y).

Occasionally, we will need to appeal to the specific definition of ordered pair, then we will revert back to our "official definition" of (x, y) as $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$.

Proof of theorem

Theorem. For sets x, y, x', y'

$$\langle x, y\rangle = \langle x', y'\rangle \;\rightarrow\; x = x' \wedge y = y'$$

Proof.

Suppose $\langle x, y\rangle = \langle x', y'\rangle$.
Case $x = y$:

$$\langle x, x\rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\} \text{ and so, } \{\{x\}\} = \{\{x'\}, \{x', y'\}\}$$

So, $\{x'\} = \{x\} = \{x', y'\}$ and thus $x' = y'$ and $x' \in \{x\}$, so $x' = x$ and $x = y'$.
Case $x \neq y$:

$$\{x\} = \{x'\} \text{ and } \{x, y\} = \{x', y'\}$$

So, $x = x'$ (first equality.) Since $y \in \{x', y'\}$ (second equality) and $y \neq x = x'$, we have $y = y'$. □

Universe of sets

Our universe of sets has expanded to include the following:

- We can begin counting:
  - $0 = \emptyset$,
  - $1 = \{0\} = \{\emptyset\}$,
  - $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$.
- The universe now includes infinitely many sets: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$, etc. (See this week's homework.)
- The axioms do not yet justify the existence of any sets with **more than two** elements.

Union

Axiom 5: Union:

$$\forall \mathcal{F} \exists A \forall x \big(\exists Y \in \mathcal{F} (x \in Y) \quad \rightarrow \quad x \in A \big)$$

- For any $\mathcal{F}$ (viewed as a *family of sets*) there is a set A which collects all *members of members* of $\mathcal{F}$. We define

$$\bigcup \mathcal{F} = \bigcup_{Y \in \mathcal{F}} Y = \{x \,|\, \exists Y \in \mathcal{F}(x \in Y)\}.$$

Justification: Let A be as in the Union axiom and apply Comprehension:

$$\bigcup \mathcal{F} = \{x \in A \,|\, \exists Y \in \mathcal{F}(x \in Y)\}.$$

- Our Union axiom is called Weak Union Axiom in H/J.

Basic definitions

Let x, y, z be sets.

- $x \cup y = \bigcup \{x, y\} = \{w \,|\, w \in x \vee w \in y\}$
- $\{x, y, z\} = \{x, y\} \cup \{z\} = \{w \,|\, w = x \vee w = y \vee w = z\}$

When $\mathcal{F} \neq \emptyset$

- $\bigcap \mathcal{F} = \bigcap_{Y \in \mathcal{F}} Y = \{x \,|\, \forall Y \in \mathcal{F}(x \in Y)\}$.
  Justification. Fix $E \in F$ and use Comprehension

$$\bigcap \mathcal{F} = \{x \in E \,|\, \forall Y \in \mathcal{F}(x \in Y)\}.$$

Note that the restriction on $\mathcal{F}$ is necessary:

- $\bigcap \emptyset = V$, and V does not exist.
- $\bigcup \emptyset = \emptyset$

Universe of sets

We can now continue counting:

Definition. The ordinal successor function is defined by $S(x) = x \cup \{x\}$ for any set x.

- $3 = S(2) = \{0, 1\} \cup \{2\} = \{0, 1, 2\}$,
- $4 = S(3) = \{0, 1, 2, 3\}$,
- $5 = S(4) = \{0, 1, 2, 3, 4\}$
- etc.

Informally, we can define

- $\mathbb{N} = \omega = \{0, 1, 2, 3, 4, 5, \ldots\}$, i.e. the set obtained by applying the operator S a *finite number of times* to 0.

This definition is circular. We will need to provide two things:

1. Guarantee the existence of a large enough set to contain all natural numbers.
2. Provide a condition φ which picks-out exactly these numbers (in a noncircular way.)

Foundation

Axiom 2: Foundation:

$$\exists y(y \in x) \quad \rightarrow \quad \exists y\big(y \in x \wedge \neg\exists z(z \in x \wedge z \in y)\big)$$

- More perspicuously,

$$x \neq \emptyset \quad \rightarrow \quad \exists y \in x(x \cap y = \emptyset)$$

- Never needed for doing ordinary mathematics.
- The main role of Foundation is to rule-out certain “pathological sets". Foundation implies the universe is neatly arranged in levels, as we will see later.

Pathological sets

Axiom 2: Foundation:

$$x \neq \emptyset \quad \rightarrow \quad \exists y \in x(x \cap y = \emptyset)$$

Foundation *rules-out* such “pathological sets" as cycles in the ϵ-relation:

- $a \in a$. In this case, $\{a\}$ is a counterexample to Foundation: $a \in \{a\}$ and $a \cap \{a\} = \{a\}$.
- $a_0 \in a_1 \in a_2 \in \ldots \in a_{n-1} \in a_0$. Then $x = \{a_0, a_1, \ldots, a_{n-1}\}$ is a counterexample to Foundation: $x \cap a_i = \{a_{i-1 \pmod n}\}$.