Math 582
Introduction to Set Theory

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Logic, the language of mathematics

- Ordinary mathematical exposition uses an informal mixture of English and logical notation.
- There is nothing “deep" about such notation: it is just a convenient abbreviation which sometimes increases clarity (and sometimes does not.)
- Our exposition of logical syntax will be semi-formal – enough detail to give a reasonably precise account of what properties are expressible in the language of set theory.
- A formal development of logical syntax is carried-out in a Logic course.

LAST, the language of set theory

- LAST (LAnguage of Set Theory) is a language suitable for describing mathematical collections (i.e. sets).
- The language will have a precisely determined set of symbols (words) and a rigid syntax (grammar). It is an example of a formal language (although my description will be informal).
- The role for this language will be to give a precise and rigorously defined concept of a definite property.
- LAST is very simple, but still sufficiently powerful enough to allow that any set we may require in mathematics is describable in LAST (i.e. the set consisting of all objects satisfying some definite property describable in LAST).

Variables

☞ We will have symbols for arbitrary sets, called variables.

$$v_0, v_1, v_2, \ldots$$

☞ A variable in mathematics is used like a pronoun is used in English. It has no fixed reference (like a name), so its meaning “floats" from context-to-context, like a pronoun.

☞ We will have no (officially sanctioned) names of sets in LAST.

Relation symbols

☞ We need to be able to make simple assertions about sets. We introduce the following relation symbols:

- Membership symbol: $\in$,
- Equality symbol: $=$.

☞ Equality has its intuitive meaning.

☞ However, we will provide axioms which will tell you all you need to know about sets and set membership in order to reason about them.

Boolean connective symbols

☞ We need to be able to combine any finite number of simple assertions to produce one big assertion. We use the Boolean connective symbols:

$$\vee, \wedge, \neg, \rightarrow, \leftrightarrow$$

☞ The intended meaning and use of these symbols mirrors corresponding English expressions (at least as mathematicians use them):

- $\vee$ means “or",
- $\wedge$ means “and"
- $\neg$ means “not"
- $\rightarrow$ means “implies" (“if-then")
- $\leftrightarrow$ means “iff" (“if and only if")

Quantifier symbols

☞ We will also require two quantifier symbols:

- $\forall$ means “for all",
- $\exists$ means “there exists"

☞ The role of the quantifiers is to provide a context to anchor the meaning of the free floating variables.

Punctuation symbols

☞ There will be two punctuation symbols to ensure that the meaning of expressions are unambiguous:

- (,) (parentheses)

Syntax

☞ An expression is simply a string of symbols of LAST. Some expressions are more significant, called formulas. Here is their definition.

(a) Any expression of the following forms are formulas of LAST.

$$v_n = v_m \qquad v_n \in v_m.$$

(b) If φ and ψ are formulas of LAST, then so are

$$(\varphi \wedge \psi) \quad (\varphi \vee \psi) \quad (\varphi \rightarrow \psi) \quad (\varphi \leftrightarrow \psi) \quad (\neg\varphi).$$

(c) If φ is a formula of LAST, then so are

$$\forall v_n\, \varphi \quad \exists v_n\, \varphi.$$

(d) Nothing else is a formula of LAST

Examples of Formulas

☞ We establish that an expression is a formula by showing how it can be constructed using the rules.

Example. $(v_1 \in v_2 \;\vee\; v_2 \in v_1)$ is a formula.

1. $v_1 \in v_2$ and $v_2 \in v_1$ are formulas by (a),
2. $(v_1 \in v_2 \;\vee\; v_2 \in v_1)$ is a formula by (b).

Example. $\forall v_0 \forall v_1\, v_0 = v_3$ is a formula.

1. $v_0 = v_3$ is a formula by (a),
2. $\forall v_1\, v_0 = v_3$ is a formula by (c),
3. $\forall v_0 \forall v_1\, v_0 = v_3$ is a formula by (c).

Unique readability

☞ Our rules of grammar for constructing formulas of LAST is unambiguous, in the following sense: there is only one way for applying the rules to construct a given formula. (This is called unique readability.)

☞ A formula ψ is called a subformula of ϕ if it occurs in ϕ. (So, ϕ is constructed by applying a rule using ψ.)

Examples of Subformulas

Example. The subformulas of $(v_1 \in v_2 \;\vee\; v_2 \in v_1)$ are

$$v_1 \in v_2,\; v_2 \in v_1,\; (v_1 \in v_2 \;\vee\; v_2 \in v_1).$$

Example. The subformulas of $\forall v_0 \forall v_1\, v_0 = v_3$ are

$$v_0 = v_3,\; \forall v_1\, v_0 = v_3,\; \forall v_0 \forall v_1\, v_0 = v_3.$$

Free vs. Bound variables

☞ Variables can be either free (floating) or bound.

- An occurrence of a variable v_n in a formula φ is free if v_n does not occur in a subformula of φ of the form $\forall v_n \psi$ or $\exists v_n \psi$. (The book uses the term parameter to mean free variable.)
- Otherwise, v_n is said to be bound

☞ A formula of LAST with no free variables is a sentence.

The sentences of LAST have a fully determinate meaning, so can be used to make assertions.

☞ A formula with free variables will have an indeterminate meaning, since its free variables will have no fixed determinate meaning.

Examples of free and bound variables

Example. $\forall v_1 (v_1 \in v_2 \rightarrow \exists v_3 v_3 \in v_2)$.

- free: v_2 (both occurrences),
- bound: v_1, v_2.

Example. $\forall v_2 (v_1 \in v_2 \rightarrow \exists v_1 v_1 \in v_2)$.

- free: v_1 (first occurrence only),
- bound: v_1 (second occurrence only), v_2 (both occurrences).

Example. Neither of the previous examples were sentences.

$\exists v_4 \forall v_3 (v_4 \in v_3 \rightarrow v_3 \in v_4)$ is a sentence.

Definite Properties

☞ We will write $\varphi(v_0, \ldots, v_n)$ to indicate that φ is a formula whose free variables, if any, are among the list $v_0, \ldots, v_n$.

☞ A definite property in LAST is any formula of the form $\varphi(v_0)$.
A definite relation in LAST is any formula of the form $\varphi(v_0, \ldots, v_n)$.

Examples.

- $v_0 \in v_1$ (the membership relation),
- $\exists v_1\, v_1 \in v_0$ (v_0 is a nonempty set),
- $v_0 \notin v_0$ (v_0 is not a member of itself),
- $\exists v_1 (v_1 \in v_0 \ \wedge\ \forall v_2 (v_2 \in v_0 \ \rightarrow\ v_2 = v_1))$ (v_0 is a singleton set).

Conventions

- We will use a wide variety of typographic symbols to stand for variables in our language:
  $x, y, z, X, Y, Z, a, b, c, A, B, C, \alpha, \beta, \gamma, \kappa, \lambda, \mathcal{F}$. The reason is that it will make it easier to read statements if you know that α is always an ordinal number, κ a cardinal number, $\mathcal{F}$ is a family of sets, etc.
- Certain variable expressions, “φ",“ψ", are not part of the language, but are used to talk about expressions of the language. They are part of the metalanguage – the language used to talk about our mathematical language.
- The text uses capital **boldface**: **P**, **A**, etc. to refer to syntactic expressions (sentences and formulas.) They write $\mathbf{P}(x)$ where I use $\varphi(x)$.
- We will use brackets: [,] as well as parentheses if this improves readability.