Math 582
Introduction to Set Theory

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Review

☞ Last time we proved

$$\mathbb{R} \hookrightarrow \mathcal{P}(\mathbb{N}) \quad \text{and} \quad \mathcal{P}(\mathbb{N}) \hookrightarrow \mathbb{R}.$$

We would like to conclude $\mathbb{R} \rightleftarrows \mathcal{P}(\mathbb{N})$.

☞ The Schröder-Bernstein Theorem provides a construction of a bijection $X \twoheadrightarrow Y$ from a pair of injections $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

☞ This is very convenient for establishing equinumerosity where producing an explicit bijection (as above) may be taxing.

Schröder-Bernstein Theorem

Theorem (Schröder-Bernstein Theorem)

For any sets A and B,

$$A \approx B \quad \longleftrightarrow \quad A \preccurlyeq B \wedge B \preccurlyeq A$$

History of the Schöder-Bernstein Theorem

From Abraham Fraenkel, *Abstract Set Theory*.

✏ The first acknowledged fully correct proof was due to Felix Bernstein in 1897. (H+J, Theorem 4.1.6.)

✏ Ernest Schröder gave a similar proof in 1897 (which Fraenkel says was “defective", although I am not sure how.)

✏ Georg Cantor conjectured the theorem true earlier and offered a proof in around this time, but the proof depended upon the comparability of size and so depended upon the Axiom of Choice. (Many references refer to the theorem as the Cantor-Bernstein-Schröder Theorem.)

✏ The earliest correct proof was due to Dedekind in 1887, but not published until 1932. Zermelo published a proof in 1908 that did not use recursion based on Dedekind’s work (although he was unaware Dedekind proved the theorem.) This proof will be the basis of a homework problem in a few weeks. ☺

Formal Definition

☞ Define $\{C_n \mid n \in \omega\}$ by recursion:

$$\begin{aligned} C_0 &= A - g[B] \\ C_{n+1} &= g\big[f[C_n]\big] \end{aligned}$$

Let $C = \bigcup_n C_n$ and $A^* = A - C$.

☞ Define the function h

$$h(x) = \begin{cases} f(x) & x \in C \\ g^{-1}(x) & x \in A^* \end{cases}$$

☞ h is well-defined on domain A (that is, g^{-1} is defined on A^*): since $A^* \subseteq A - C_0 \subseteq g[B]$.

Verification: Surjectivity

Surjectivity. Let $b \in B$.

☞ Suppose $b \in f[C]$. Then, there is some $a \in C$ with $f(a) = b$. In this case, $h(a) = f(a) = b$.

☞ Suppose $b \notin f[C]$. If $g(b) \in C$, then for some n,

$$g(b) \in C_{n+1} = g\big[f[C_n]\big]$$

So, $g(b) = g(f(z))$ for some $z \in C_n$.
Since g is injective, $b = f(z) \in f[C_n]$ ↯.

✓ So, $g(b) \in A^*$ and $h(g(b)) = g^{-1}(g(b)) = b$.

Verification: Injectivity

Injectivity. Suppose that $h(a) = h(c)$. Since f injective on C and g injective on A^* we must have either (i) $a = c$ or (ii) $c \in C$ and $a \in A^*$ (or vice-versa.)

☞ Suppose (ii) $c \in C_n$ for some n and $a \in A^*$, Then

$$f(c) = h(c) = h(a) = g^{-1}(a)$$

☞ $c \in C_n$ implies $g(f(c)) \in g[f[C_n]] = C_{n+1} \subseteq C$. So,

$$a = g(g^{-1}(a)) = g(f(c)) \in C_{n+1} \subseteq C \lightning$$

✓ Therefore, (i) $a = c$, so that h is injective.

Elements of Naive set theory

☞ Naive set theory starts with the existence of certain sets of mathematical objects:

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

as well as

- Ordered pairs, and more generally ordered n-tuples,
- Functions

☞ On top of this it makes two assumptions about sets

- **Naive Comprehension Principle** (**NCP**),
- **Extensionality**

Basic assumptions of Naive set theory

The two main principles of naive set theory:

- (**Naive Comprehension Scheme**) For each n-ary definite property, **P**, there is a set

$$A = \{\overline{x} \mid \mathbf{P}(\overline{x})\}.$$

whose members are precisely all the n-tuples of objects $\overline{x} = (x_1, \ldots, x_n)$ which satisfy **P**:

$$\overline{x} \in A \leftrightarrow \mathbf{P}(\overline{x})$$

- (**Extensionality**) For all sets A and B

$$\forall z(z \in A \leftrightarrow z \in B) \rightarrow A = B$$

Naive set theory

☞ It is necessary to restrict (**NCP**) to avoid questions of vagueness.

Clarification. A condition **P** is definite if for each n-tuple of objects $\overline{x} = (x_1, \ldots, x_n)$, it is determined unambiguously whether $\mathbf{P}(\overline{x})$ is true or false.

☞ Any property or relation of the basic mathematical objects are definite.

We also assume that any property which can be logically built from the relations

$$\begin{aligned} x \in y &\longrightarrow x \text{ is a member of } y \\ x = y &\longrightarrow x \text{ is identical to } y \\ Set(x) &\longrightarrow x \text{ is a set .} \end{aligned}$$

is a definite property.

Some sets definable

- $\emptyset = \{x \mid x \neq x\}$, the empty set.
- $V = \{x \mid Set(x) \ \wedge \ x = x\}$, the universal set of sets.
- $S = \{x \mid x \in x\}$. This set is not empty since $V \in S$.
- $E = \{(x, y) \mid x \in y\}$,
- $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$, for any sets A, B
- $A \cap B = \{x \mid x \in A \ \wedge \ x \in B\}$, for any sets A, B
- $A^c = \{x \mid x \notin A\}$, for any set A,
- $\{a\} = \{x \mid x = a\}$, for any object a
- $\{a, b\} = \{x \mid x = a \vee x = b\}$, for any objects a, b
- $\mathcal{P}(A) = \{x \mid x \subseteq A\} = \{x \mid \forall y(y \in x \ \rightarrow \ y \in A\}$, for any set A.

Sets are all there are

☞ By 1900, it was known that starting from only sets, together with the **Naive Comprehension Principle**, you could construct the objects:

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

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    - $0 = \emptyset = \{x \mid x \neq x\}$
    - $1 = \{x \mid \exists! y(y \in x)\} = \{x \mid \exists y(y \in x \wedge \forall z(z \in x \rightarrow z = y)\}$
    - $2 = \{x \mid x$ is a two-element set $\}$
    - $3 = \{x \mid x$ is a three-element set $\}$, etc.
- $\mathbb{N} = \{x \mid x = 0 \vee x = 1 \vee x = 2 \vee x = 3 \vee \ldots\}$??
  (Frege and Dedekind figured-out how to replace "..." with a definite condition.)
- $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are constructed from $\mathbb{N}$ using (**NCP**) in the standard way in any analysis course.

Sets are all there are

☞ By 1900, it was known that you could define ordered pairs and functions using only sets and the **Naive Comprehension Principle**:

- Ordered pairs: one way of doing this is define

$$(a, b) \leftrightarrow \{\{0, a\}, \{1, b\}\}$$

  There are other ways as well. From ordered pairs we get n-tuples.
- Functions: defined as sets of ordered pairs.

☞ So, by 1900 it was known that the only mathematical objects we need are sets, together with the assumptions about sets:

- **Naive Comprehension Principle** (**NCP**),
- **Extensionality**

☞ Unfortunately, (**NCP**) is inconsistent.

The Russell set

☞ By (**NCP**), the Russell set exists:

$$R = \{x \mid Set(x) \ \wedge \ x \notin x\}.$$

So,

$$\forall x \big(x \in R \quad \leftrightarrow \quad Set(x) \ \wedge \ x \notin x\big)$$

☞ Since R is a set it is an object, so the quantifier $\forall x$ applies to R:

$$R \in R \leftrightarrow R \notin R \ E$$

☞ Therefore, (**NCP**) is inconsistent.

Responses to the Paradoxes

☞ By 1901 (when Russell first revealed the paradox of the Russell set), the mathematical world was aware of the deep seeded difficulties with the naive use of sets.

☞ However, the powerful tools unleashed by Cantor for studying infinite sets was too important to give-up.

☞ Thus, David Hilbert wrote (in 1926)

> *From the paradise which Cantor created, no one shall be able to expel us.*

☞ The challenge is to fill the hole left by the failure of the **Naive Comprehension Principle**.

Responses to the Paradoxes

- Cantor began investigating infinite sets in 1870s. His work led to a number of counter-intuitive results, and his techniques met much resistance among important mathematicians of the time (such as Kronecker, Poincare, Brouwer, Borel, Lebesgue.)
- Russell first revealed his paradox in a letter to Frege in 1902. The revelation created a foundational crisis. (Cantor, himself, had been aware for at least ten years prior of the possibility of paradox by unrestricted use of (**NCS**).)
- In 1908 Zermelo published the first axiomatization of set theory (essentially ZC^-.) His motivation was two fold
  - To isolate exactly what instances of (**NCS**) were needed to develope mathematics, and formulate these as axioms about sets.
  - To prove the consistency of the axioms. (Which he never accomplished.)
- In 1922 Skolem proposed two additions to Zermelo's axioms (based on Fraenkel's work): Foundation and Replacement. This is the system *ZFC* commonly accepted today.