Math 582
Introduction to Set Theory

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Review of two (uncountable) sets

☞ The set of infinite binary sequences

$$\Delta = \{(b_0, b_1, \ldots) \mid \forall i \left[b_i = 0 \;\vee\; b_i = 1 \right]\}$$

☞ The powerset of the natural numbers:

$$\mathcal{P}(\mathbb{N}) = \{X \mid X \subseteq \mathbb{N}\}$$

Orders of Infinity

☞ We have seen several examples of infinite countable sets:

$$\mathbb{N} \approx \mathbb{Z} \approx \mathbb{Q}.$$

☞ We have produced at least two "orders of infinity":

$$\mathbb{N} \prec \Delta \preccurlyeq \mathbb{R}.$$

☞ In the next section we will show:

$$\mathbb{R} \approx \Delta \approx \mathcal{P}(\mathbb{N}).$$

Thus, we have only two "orders of infinity", $\mathbb{N}$ and $\mathbb{R}$ – so far.

Infinite binary sequences vs. $\mathcal{P}(\mathbb{N})$

Lemma. $\mathcal{P}(\mathbb{N}) \approx \Delta$.

Since $\Delta \preccurlyeq \mathbb{R}$ (from the proof that $\mathbb{R}$ is uncountable):
Corollary. $\mathcal{P}(\mathbb{N}) \preccurlyeq \mathbb{R}$.

Proof

Proof.
Consider the map from $\mathbb{N}$ to Δ given by $(A \mapsto \chi_A)$ where $\chi_A : \mathbb{N} \to \{0, 1\}$ is given by

$$\chi_A(n) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A. \end{cases}$$

☞ The map is injective:
if $A \neq B$ then for any n which belongs to only one of the sets A, B, we have $\chi_A(n) \neq \chi_B(n)$.
☞ The map is surjective:
For any sequence $\delta \in \Delta$, let

$$A_\delta = \{n \in \mathbb{N} \,|\, \delta(n) = 1\}.$$

Then, $A_\delta \mapsto \delta = \chi_{A_\delta}$.
q.e.d.

$\mathbb{R}$ vs. $\mathcal{P}(\mathbb{N})$

We now show:
Lemma. $\mathbb{R} \preccurlyeq \mathcal{P}(\mathbb{N})$.

Corollary. $\mathbb{R} \approx \Delta \approx \mathcal{P}(\mathbb{N})$.

The corollary is a direct consequence of the following:
Schröder-Bernstein Theorem. For any two sets A and B,

$$A \preccurlyeq B \,\wedge\, B \preccurlyeq A \quad \to \quad A \approx B.$$

Proof will be given next lecture.

Proof of Lemma

Proof. Proof uses Dedekind cuts.

☞ It is enough to show $\mathbb{R} \preccurlyeq \mathcal{P}(\mathbb{Q})$ since $\mathbb{N} \approx \mathbb{Q}$, so that $\mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{Q})$. (See HW3 for proof).

☞ Define the function π as follows (for each $x \in \mathbb{R}$)

$$\pi(x) = \{q \in \mathbb{Q} \,|\, q < x\} \subseteq \mathbb{Q}.$$

☞ π is injective:

if $x < y$ then there is some rational q between them: $x < q < y$, and hence $q \in \pi(y) - \pi(x)$.

q.e.d.

More orders of infinity

☞ We have seen two “orders of infinity" – $\mathbb{N}$ and $\mathbb{R}$. There are many others.

Theorem (Cantor)
For every set A,

$$A \prec \mathcal{P}(A).$$

Proof of Theorem

Proof.

☞ That $A \preccurlyeq \mathcal{P}(A)$ follows from the fact that the function

$$\left(x \mapsto \{x\}\right)$$

is an injection.

☞ To show $A \prec \mathcal{P}(A)$ we use a generalization of Cantor's second diagonalization method, which we used to show that the set of infinite binary sequences Δ is uncountable.

Diagonalization

☞ Suppose (for contradiction) that there is a surjection $\pi : A \twoheadrightarrow \mathcal{P}(A)$.

☞ Define the set

$$R = \{x \in A \mid x \notin \pi(x)\}.$$

Since $R \subseteq A$ and π is a surjection, $R = \pi(x)$ for some $x \in A$.

☞ Suppose $x \in R = \pi(x)$. Then $x \in \pi(x)$, but also $x \notin \pi(x)$ by the definition of R. This is impossible. So, $x \notin R$.

☞ Thus, $x \notin R = \pi(x)$. But then, $x \in R$ by the definition of R, which is impossible.

☞ Therefore, there can be no surjection $\pi : A \twoheadrightarrow \mathcal{P}(A)$, and so $A \prec \mathcal{P}(A)$.

q.e.d.

Transfinite Arithmetic

☞ The classification and study of these orders of infinity is one of the most important tasks in modern set theory arising from the work of Cantor in the late nineteenth century.

☞ Transfinite arithmetic introduces and studies the operations of addition, multiplication and exponentiation on infinite numbers, which extends these operations on finite numbers.

Two problems

☞ At the end of the nineteenth century there remained two fundamental problems on equinumerosity that remained unsolved.

We state them in the form presented by Cantor (as hypotheses).

Problem 1. Hypothesis of Cardinal Comparability.
For any two sets A and B, either $A \preccurlyeq B$ or $B \preccurlyeq A$.

Problem 2. Continuum Hypothesis.
There is no set of real numbers X with cardinality intermediate between those of $\mathbb{N}$ and $\mathbb{R}$:

$$(\textbf{CH}) \qquad \forall X \subseteq \mathbb{R} \left[X \preccurlyeq \mathbb{N} \ \vee \ X \approx \mathbb{R} \right].$$

CH is a special case of the Generalized Continuum Hypothesis, the statement that for every infinite set A,

$$(\textbf{GCH}) \qquad \forall X \subseteq \mathcal{P}(A) \left[X \preccurlyeq A \ \vee \ X \approx \mathcal{P}(A) \right].$$

Problem 1: Cardinal comparability

☞ Cantor was never able to produce a proof of **Problem 1**, although he announced a solution to the problem in 1895 and produced a sketch for Dedekind in 1899. (This last wasn't published until 1932, and the argument had problems.)

☞ The first resolution to the problem was given by Ernst Zermelo in 1904, and we will come back to it when we discuss the Axiom of Choice.

☞ The **Hypothesis of Cardinal Comparability** is equivalent to Zermelo's Axiom of Choice, as well as the Well-orderability Principle: every set can be well-ordered.

Problem 2: The Continuum Hypothesis

☞ The resolution of **CH** was the first of Hilbert's famous 23 open problems (1900).

☞ We will look at what is known about this problem at the end of class. However, it is a consequence of work by Kurt Gödel (1938) and Paul Cohen (1964), that this problem is unresolvable in Zermelo-Frankel Set Theory (ZFC). We will begin our study in ZFC next week.

☞ Hilbert's second open question was to establish the Consistency of Peano Arithmetic (using finite methods). Gödel (1931) and Turing (1936) showed that there is no resolution to this problem. We will study the (second-order) version of Peano Arithmetic in several weeks.

Opening to Hilbert's 23 problems

☞ The final paragraph of Hilbert's lecture (1900) has a refreshing conclusion:

> *The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena. That is may completely fufil this high mission, may the new century bring it gifted masters and many zealous and enthusiastic disciples.*