Math 582
Introduction to Set Theory

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Finite Cartesian products

Lemma. If $A_1, \ldots, A_n$ are all countable, so is their Cartesian product $A_1 \times \ldots \times A_n$.

Proof

Proof. By induction on n.

Basis case. For $n = 2$: Let A and B be countable. If either is the empty, then $A \times B$ is the empty, so countable. Suppose neither is empty, and write B as

$$B = \{b_0, b_1, \ldots\};$$

then we can write $A \times B$ as:

$$\bigcup_{n=0}^{\infty} (A \times \{b_n\}).$$

Since

$$A \approx A \times \{b_n\} \qquad \text{by } \big(x \mapsto (x, b_n)\big),$$

$A \times B$ is a countable union of countable sets, so countable.

Proof – continued

Inductive step. Let $A_1, \ldots, A_n, A_{n+1}$ be countable sets and suppose that $A_1 \times \ldots \times A_n$ is countable (inductive hypothesis).

Recall that

$$A_1 \times \ldots \times A_n \times A_{n+1} = \big(A_1 \times \ldots \times A_n\big) \times A_{n+1}.$$

Since $A_1 \times \ldots \times A_n$ is countable and A_{n+1} is countable, it follows that $A_1 \times \ldots \times A_n \times A_{n+1}$ is countable by the Basis case.

q.e.d.

Finite sequences

Corollary. For every countable set A, the union

$$\bigcup_{n=2}^{\infty} A^n = \{(x_1, \ldots, x_n) \,|\, n \geq 2 \ \wedge \ x_1, \ldots, x_n \in A\}$$

is countable.

Proof. The union is a countable union of countable sets (by the previous Lemma), so is countable.

Terminology. We call the union

$$\bigcup_{n=2}^{\infty} A^n.$$

the set of finite sequences from A. (This is formally defined in H+J in Section 3.3.)

Uncountable sets

☞ Cantor showed the existence of uncountable sets.
The method of proof introduces Cantor's second diagonalization method:

Theorem. The set of infinite binary sequences

$$\Delta = \{(b_0, b_1, \ldots) \,|\, \forall i \, [b_i = 0 \ \vee \ b_i = 1]\}$$

is uncountable.

Proof

Proof. Suppose (towards a contradiction) that Δ is countable, so that there is an enumeration

$$\Delta = \{\beta_0, \beta_1, \ldots\},$$

where for each n,

$$\beta_n = (b_0^n, b_1^n, \ldots).$$

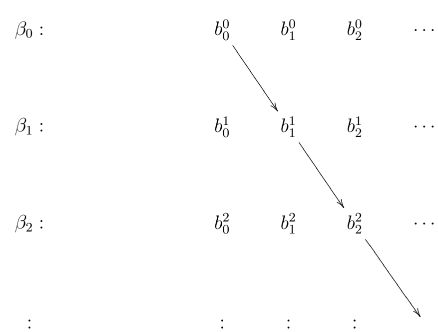
☞ We construct a table with these sequences (as in the first diagonalization method from Lecture 4), and define a new sequence δ by interchanging 0 and 1 along the diagonal sequence $b_0^0, b_0^1, \ldots$:

$$\delta(n) = 1 - b_n^n.$$

See the next slide.

Proof – continued

☞ The diagonal sequence δ is defined by $\delta(n) = 1 - b_n^n$:



Proof – continued

☞ Then $\delta \neq \beta_n$ for every n, since

$$\delta(n) = 1 - b_n^n = 1 - \beta_n(n) \neq \beta_n(n).$$

☞ So, the sequence $\beta_0, \beta_1, \ldots$ does not enumerate the entire set Δ, contrary to our hypothesis.

q.e.d.

The reals are uncountable

Theorem. The set $\mathbb{R}$ of real numbers is uncountable.

☞ I will give Cantor's construction from 1883. He used the construction of the Cantor set to show the existence of a perfect set that is nowhere dense

- A perfect set is a closed set without isolated points.
- A set is nowhere dense if it contains no open set (i.e. has empty interior).

Intuitively, the Cantor set is densely packed, but with interior.

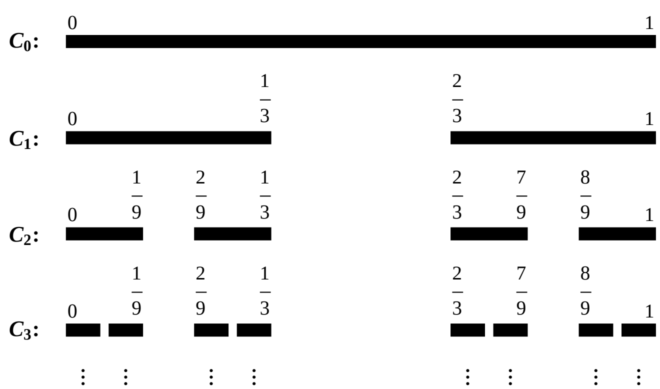
Proof

Step 1. We first construct a sequence of sets $\mathcal{C}_0, \mathcal{C}_1, \ldots$ of real numbers, by recursion, which satisfies the following three conditions.

1. $\mathcal{C}_0 = [0,1]$, where $[a,b] = \{r \mid a \leq r \leq b\}$.
2. Each $\mathcal{C}_n$ is a union of 2^n closed disjoint intervals and
$$\mathcal{C}_0 \supseteq \mathcal{C}_0 \supseteq \ldots \supseteq \mathcal{C}_n \supseteq \mathcal{C}_{n+1} \supseteq \ldots.$$
3. $\mathcal{C}_{n+1}$ is constructed by removing the middle third of each interval in $\mathcal{C}_n$.
Replace each $[a,b]$ in $\mathcal{C}_n$ by the two disjoint closed intervals:
$$\begin{aligned} L[a,b] &= [a, a + \frac{1}{3}(b-a)], \\ R[a,b] &= [a + \frac{2}{3}(b-a), b] \end{aligned}$$
The length of each interval in $\mathcal{C}_n$ is 3^{-n}.

The Cantor set

☞ The first four stages of the Cantor set construction:



Proof – continued

Step 2. With each infinite binary sequence $\delta \in \Delta$ we associate a sequence of closed intervals

$$F_0^\delta, F_1^\delta, \ldots,$$

by the following recursion:

$$\begin{aligned} F_0^\delta &= [0,1] \\ F_{n+1}^\delta &= \begin{cases} LF_n^\delta & \text{if } \delta(n) = 0, \\ RF_n^\delta & \text{if } \delta(n) = 1. \end{cases} \end{aligned}$$

We can prove the following properties by induction on n:

(a) F_n^δ is a closed interval from $\mathcal{C}_n$,
(b) The length of F_n^δ is 3^{-n},
(c) The intervals are decreasing:

$$F_0^\delta \supseteq F_1^\delta \supseteq \ldots,$$

Proof – continued

Step 3. We associate a real number to each infinite binary sequence δ, as follows.

By properties (a) - (c) for the F_n^δ, it follows that the intersection

$$\bigcap_{n=0}^{\infty} F_n^\delta \neq \emptyset$$

and in fact contains a single real number.
This is by the **completeness property** of the reals.

Define

$$f(\delta) = \text{ the unique real in } \bigcap_{n=0}^{\infty} F_n^\delta,$$

Proof – completed

Then f maps the uncountable set Δ into the Cantor set

$$\bigcap_{n=0}^{\infty} \mathcal{C}_n \subseteq [0,1].$$

Step 4. It only remains to show that f is injective. Suppose $\delta \neq \varepsilon$, and let n be least such that $\delta(n) \neq \varepsilon(n)$. We suppose $\delta(n) = 0$ for definiteness.

☞ We have $F_n^\delta = F_n^\varepsilon$ (by the choice of n) and

$$F_{n+1}^\delta = LF_n^\delta \qquad F_{n+1}^\varepsilon = RF_n^\varepsilon = RF_n^\delta.$$

But, $LF_n^\delta \cap RF_n^\delta = \emptyset$. Since,

$$f(\delta) \in LF_n^\delta \quad \text{and} \quad f(\varepsilon) \in RF_n^\varepsilon = RF_n^\delta,$$

we indeed have $f(\delta) \neq f(\varepsilon)$.

q.e.d.