# Math 582
# Introduction to Set Theory

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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## Equinumerous

**Definition**. Two sets $A$ and $B$ are equinumerous or equal in cardinality iff there is a bijection between their elements. We write

$$A \approx B \leftrightarrow \exists f \left[ f : A \leftrightarrows B \right].$$

**Note on terminology**. The material in this lecture corresponds to Chapter 4.1-3 of H+J. They use the term equipotent, where I am using equinumerous.

## Equinumerous

☞ No finite set can be equinumerous with a proper subset; however, this is not true of infinite sets.

**Example.**

$$\mathbb{N} = \{0, 1, 2, \ldots\} \approx \{1, 2, 3, \ldots\}$$

via the correspondence

$$(x \mapsto x + 1)$$

**Example.** In the real numbers,

$$(0, 1) \approx (0, 2) \qquad \text{where } (p, q) = \{r \in \mathbb{R} \mid p < r < q\},$$

via the correspondence

$$(x \mapsto 2x).$$





## Equivalence relation

☞ Equinumerosity is an equivalence relation between sets:

**Proposition.** For all sets $A, B, C$,

$$\begin{aligned} A &\approx A, \\ A \approx B &\rightarrow B \approx A, \\ A \approx B \wedge B \approx C &\rightarrow A \approx C \end{aligned}$$

## Comparison of size

**Definition.** The set $A$ is less than or equal to $B$ in size iff it is equinumerous with a subset of $B$. We write

$$A \preceq B \;\leftrightarrow\; \exists C\,[C \subseteq B \;\wedge\; A \approx C].$$





## Proposition

**Proposition.** For all sets $A$ and $B$

$$A \preceq B \;\leftrightarrow\; \exists f\,[f : A \hookrightarrow B]$$

**Proposition.** For all sets $A, B, C$

$$\begin{array}{c} A \preceq A, \\ A \preceq B \;\wedge\; B \preceq C \quad\rightarrow\quad A \preceq C \end{array}$$

**Note.** It is also true that $\preceq$ is an ordering relation

$$A \preceq B \;\wedge\; B \preceq A \;\rightarrow\; A \approx B.$$

However, this is a difficult result known as the Schröder-Bernstein theorem, which we will prove later.

## Finite sets

**Definition.** A set $A$ is finite if there exists a natural number $n$ such that

$$A \approx \{i \mid i < n\} = \{0, 1, \dots, n-1\};$$

otherwise, $A$ is finite.

**Example.**

- The empty set $\emptyset$ is finite since $\emptyset \approx \{i \mid i < 0\}$.
- Any singleton set, $\{x\}$, is finite since $\{x\} \approx \{i \mid i < 1\}$.





## Countable sets

**Definition.** A set is countable (or denumerable) if it is either finite or equinumerous with the set of natural numbers $\mathbb{N}$; otherwise, it is uncountable.

**Proposition.** A nonempty set $A$ is countable iff $A$ has an enumeration, a surjection $\pi : \mathbb{N} \twoheadrightarrow A$, so that

$$A = \{\pi(0), \pi(1), \dots\}.$$

## Proof: $\longrightarrow$

**Proof**. Suppose *A* is countable.

☞ If *A* is infinite, then there is a bijection $\pi : \mathbb{N} \rightleftarrows A$ by definition.

☞ If *A* is finite and nonempty, then we have a bijection $f : \{i \mid i < n\} \leftrightarrows A$ for some $n > 0$. Define

$$\pi(i) = \begin{cases} f(i) & \text{if } i < n, \\ f(0) & \text{if } i \geq n. \end{cases}$$

Then $\pi : \mathbb{N} \twoheadrightarrow A$ is an enumeration of *A*.

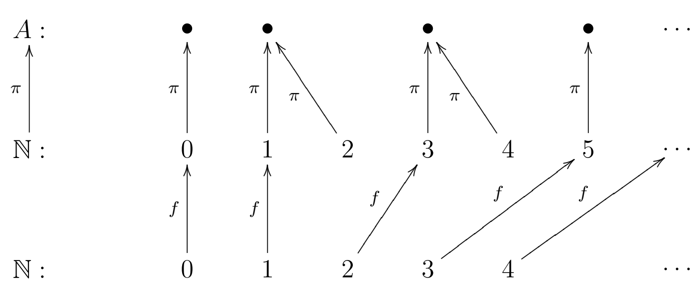




## Proof: $\longrightarrow$

Conversely, suppose *A* has an enumeration $\pi : \mathbb{N} \to A$, but is not finite.

$\pi$ may fail to be a bijection because of repetitions: $\pi(i) = \pi(j)$ but $i \neq j$. We define a bijection $f : \mathbb{N} \rightleftarrows A$ by skipping repetitions.



## Proof: $\rightarrow$

☞ Since $A$ is not finite, for every finite set $\{a_0, a_1, \ldots, a_n\}$ of $A$, there exists some $m \in \mathbb{N}$ with $\pi(m) \notin \{a_0, a_1, \ldots, a_n\}$.

Define $f$ by recursion as follows:

$$\begin{aligned} f(0) &= \pi(0), \\ f(n+1) &= \pi(m) \\ &\quad \text{where } m > n \text{ is least with } \pi(m) \notin \{f(0, f(1), \ldots, f(n)\}. \end{aligned}$$

☞ It is obvious that $f$ is injective, so we show it is surjective.

Let $x \in A$, so that $x = \pi(n)$ for some $n$. If $x \in \{f(0), \ldots, f(n-1)\}$ we are done, otherwise $f(n) = \pi(n)$ by definition of $f$.

**q.e.d.**





## Countable unions of countable sets

The next result is one of the most basic results in counting. It uses Cantor's first diagonal method.

**Theorem**. (Cantor) For each sequence $A_0, A_1, \ldots$ of countable sets, the union

$$A = \bigcup_{n=0}^{\infty} A_n$$

is also a countable set.

## Proof of theorem

**Proof**. WLOG (with loss of generality) we may assume that none of the sets $A_n$ is empty.

☞ Let $\pi^n : \mathbb{N} \twoheadrightarrow A_n$ be an enumeration. We write

$$a_i^n = \pi^n(i)$$

for simplicity, so that for each $n$

$$A_n = \{a_0^n, a_1^n, \ldots\}$$



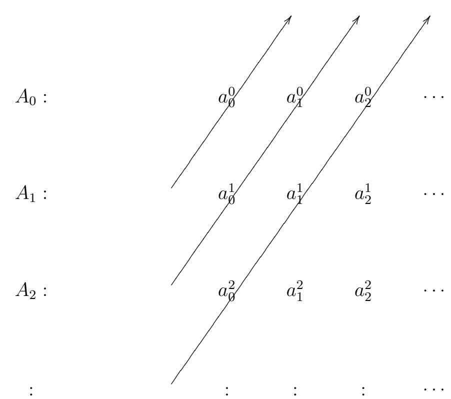


## Proof of theorem – continued

☞ Enumerate $A$ by following the arrows in the picture:

$$A = \{a_0^0, a_0^1, a_1^0, a_0^2, a_1^1, a_2^0, \ldots\}.$$

**q.e.d.**



## Integers are countable

**Corollary**. The set of integers $\mathbb{Z}$ is countable.

**Proof**. $\mathbb{Z} = \mathbb{N} \cup \{-1, -2, \ldots\}$ and the set of negative integers is countable by the correspondence

$$\big(x \mapsto -(x+1)\big)$$





## Rationals are countable

**Corollary**. The set of rationals $\mathbb{Q}$ is countable.

**Proof**. The set of positive rationals $\mathbb{Q}^+$ is countable because it is the countable union of countable sets:

$$\mathbb{Q}^+ = \bigcup_{n=1}^{\infty} \{\frac{m}{n} \mid m \in \mathbb{N}\}.$$

Similarly, the negative rationals $\mathbb{Q}^-$ is countable.
Finally, express $\mathbb{Q}$ as

$$\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+,$$

a countable union of countable sets. So, $\mathbb{Q}$ is countable.