Math 582
Intro to Set Theory
Lecture 35

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Definition: Stationary Sets

☞ Recall the convention that κ is an uncountable regular cardinal.

Definition

A set $S \subseteq \kappa$ is stationary in κ if and only if $S \cap C \neq \emptyset$ for every club $C \subseteq \kappa$.

Equivalently, S is stationary iff $\kappa - S$ does not contain a club in κ.

☞ Every club is stationary. However, not every stationary set is a club:

$$S = \kappa - \{\omega\}$$

is not closed, but it is clearly stationary.

Simple facts about stationary sets

1. Stationary sets are unbounded.
   Reason. For each $\alpha < \kappa$,
   $$C_\alpha = \kappa - \alpha$$
   is a club. If S is stationary, then $S \cap C_\alpha \neq \emptyset$, so that S contains an ordinal bigger than α.
2. If A is stationary and C is a club, then $A \cap C$ is stationary.
   Reason. If D is any club, then $C \cap D$ is a club, so that $(A \cap C) \cap D$ is nonempty.
3. If A is stationary in κ then the following set is stationary:
   $$A \cap \{\text{limit ordinals of } \kappa\}$$
   Reason. The set of limit ordinals in κ is a club.

Examples of Stationary Sets

☞ The following produces many examples of stationary sets.
For example, if $\kappa = \omega_2$, then

$$\begin{aligned} S_\omega &= \{\alpha < \omega_2 \,|\, \text{cf}(\alpha) = \omega\} \\ S_{\omega_1} &= \{\alpha < \omega_2 \,|\, \text{cf}(\alpha) = \omega_1\} \end{aligned}$$

are stationary but not clubs (they are disjoint).

Lemma

For each regular cardinal $\lambda < \kappa$, the set

$$S_\lambda = \{\alpha < \kappa \,|\, cf(\alpha) = \lambda\}$$

is a stationary set.

Proof

Let $\lambda < \kappa$ a regular cardinal, and consider

$$S_\lambda = \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}.$$

☞ Let $f : \kappa \to \kappa$ be a normal function. Then

$$f(\lambda) = \sup_{\xi<\lambda} f(\xi),$$

so that the cofinality of $f(\lambda)$ is λ by the regularity of λ.
Thus, $f(\lambda) \in S_\lambda$, since the range of f is a club.

☞ Since S_λ meets the range of every normal function, S_λ meets every club, and is thus stationary.

Examples of Nonstationary Sets

☞ Any set $X \subseteq \kappa$ which is bounded is nonstationary.

☞ The set

$$\{\alpha + 1 \mid \alpha < \kappa\}$$

is nonstationary, since the set of limit ordinals is a club in κ.
For the same reason, the set of successor ordinals in κ is also nonstationary.

☞ Let $\mathrm{NS}(\kappa)$ be the family of nonstationary sets of κ. Then $\mathrm{NS}(\kappa)$ is an ideal, and in fact κ complete.
The ideal $\mathrm{NS}(\kappa)$ is the dual of the filter of κ clubs, $\mathcal{F}_\kappa^{\clubsuit}$.

Definition: Regressive functions

Definition

Let $S \subseteq \kappa$. A function $f : S \to \kappa$ is said to be regressive if and only if $f(\alpha) < \alpha$ for every nonzero $\alpha \in S$.

☞ The 'quintessential' regressive function is the predecessor function on the natural numbers:

$$f(0) = 0 \qquad f(n+1) = n.$$

However, ω is unusual for allowing increasing regressive functions (which are not eventually constant.)

Definition: Regressive functions

☞ The name 'stationary' set came from the following characterization of these sets in terms of regressive functions.

Theorem

Let $S \subseteq \kappa$. The following are equivalent.

(i) *S is stationary.*

(ii) *If $f : S \to \kappa$ is regressive, then f is constant on a stationary set: $f^{-1}[\{\gamma\}]$ is stationary in κ for some $\gamma < \kappa$.*

(iii) *If $f : S \to \kappa$ is regressive, then f is constant on a set of cardinality κ: $\left|f^{-1}[\{\gamma\}]\right| = \kappa$ for some $\gamma < \kappa$.*

Note. Regressive functions were first introduced by Fodor in 1953, where they were called push-down functions. This theorem was called the Push-Down Theorem.

Proof

$(i) \Rightarrow (ii)$. Let S be a stationary set and $f : S \to \kappa$ a regressive function. Suppose that $f^{-1}[\{\gamma\}]$ is not stationary, for any $\gamma < \kappa$.
For each γ fix C_γ with $f^{-1}[\{\gamma\}] \cap C_\gamma = \emptyset$ and let $C = \Delta_{\gamma<\kappa} C_\gamma$.

☞ Since $C \neq \emptyset$, take any $\alpha \in C$, so that $\alpha \in C_\gamma$ for each $\gamma < \alpha$. But f is regressive, $f(\alpha) = \gamma < \alpha$, and so

$$\alpha \in f^{-1}[\{\gamma\}] \cap C_\gamma \quad \lightning$$

Thus, for some γ the set $f^{-1}[\{\gamma\}]$ is stationary.

$(ii) \Rightarrow (iii)$. Trivial, since stationary sets have cardinality κ.
S is unbounded in κ and κ is regular.

Proof – continued

$(iii) \Rightarrow (i)$. Prove the contrapositive. Assume $\neg(i)$: S is not stationary. We will produce a regressive function $f : S \to \kappa$ with no unbounded $f^{-1}[\{\gamma\}]$. Clearly, if S is bounded in κ, $\neg(iii)$ must be true. Suppose S is unbounded in κ.

☞ Let C be a club with $C \cap S = \emptyset$. Define $f : S \to \kappa$ for each $\alpha \in S$ by

$$f(\alpha) = \sup C \cap \alpha.$$

Since C is a club and $C \cap S = \emptyset$, $f(\alpha) < \alpha$. So, f is regressive.

☞ f is increasing by definition, and since C is unbounded, so is f. Thus, for any $\gamma < \kappa$, the set $f^{-1}[\{\gamma\}]$ is bounded and so has cardinality less than κ.
This establishes $\neg(iii)$.

Disjoint Stationary Sets

☞ We saw earlier that the following sets are stationary for each regular $\lambda < \kappa$:

$$S_\lambda = \{\alpha < \kappa \,|\, \mathrm{cf}(\alpha) = \lambda\}$$

These will be disjoint stationary sets when $\kappa > \omega_1$.

☞ When $\kappa = \omega_1$ it is not obvious that ω_1 can be split into even two disjoint stationary sets. H+J provide a clever argument that this is possible in Example 11.3.12 on page 211.

Robert Solovay has gone far beyond this (extending work of Fodor):

Theorem (Solovay 1971, Fodor 1966)

Let κ be an uncountable regular cardinal and $A \subseteq \kappa$ stationary in κ. Then A can be split as the union of κ many, pairwise disjoint sets stationary in κ.

Solovay for successors

☞ I will prove Solovay's theorem in the case of successor cardinals.
Recall that regular limit cardinals are inaccessibles, whose existence cannot be proven in ZFC. Solovay's proof in this case requires some tweaks to the successor case.

Theorem

Let $\kappa > \omega$ and let $A \subseteq \kappa^+$ be a set stationary in κ^+. Then A can be represented as the union of κ^+ many, pairwise disjoint sets stationary in κ^+.

Definition: Almost bounded functions

☞ Let κ be a regular uncountable cardinal. Given $A \subseteq \kappa$ and a property $\Phi(x)$, we will say:

- For almost all $\alpha \in A$ $\Phi(\alpha)$ holds.

when the set $\{\alpha \in A \,|\, \Phi(\alpha) \text{ fails }\} \in \mathtt{NS}(\kappa^+)$ (i.e. nonstationary, or "insignificant").

Definition. Let κ be a regular uncountable cardinal and $A \subseteq \kappa$.
A function f is almost bounded on A if there is an ordinal $\rho < \kappa$ such that

- $f(\alpha) < \rho$, for almost all $\alpha \in A$.

☞ Intuitively, the ideal of nonstationary sets $\mathtt{NS}(\kappa)$ are "small" or "inconsequential". So, an almost bounded function is one which is bounded, except on an inconsequential set.

Lemma on Almost bounded functions

☞ The importance of almost bounded functions is explained in the next lemma (where λ need not be a successor cardinal.)

Lemma

Let λ be a regular uncountable cardinal and $A \subseteq \lambda$ stationary.
If there is a regressive function f on A which is NOT almost bounded, then A is the union of λ pairwise disjoint stationary subsets.

Proof of Lemma

☞ Let $A \subseteq \lambda$ be stationary and f a regressive function on A which is not almost bounded. For each $\alpha < \lambda$, let

$$A_\alpha = f^{-1}[\{\alpha\}] \cap A \quad \text{and} \quad M = \{\alpha \mid A_\alpha \text{ is stationary. }\}$$

The sets A_α with $\alpha \in M$ are disjoint and stationary, so it is sufficient to show that $|M| = \lambda$.

☞ We show M is cofinal in λ. Fix $\beta < \lambda$. Since f is NOT almost bounded, the following set is stationary

$$B = \{\xi \in A \mid f(\xi) \geq \beta\}.$$

But f is regressive in A, so regressive in B, and thus for some $\alpha \geq \beta$

$$f^{-1}[\{\alpha\}] \cap B \subseteq A_\alpha$$

is stationary.

Proof of Theorem

We continue with the proof of Solovay's partition theorem.

☞ We may assume that for the stationary set A that

$$A \subseteq (\kappa, \kappa^+) \cap \{\text{limit ordinals of } \kappa^+\}$$

Note that this restriction depends on κ^+ being a successor.

☞ Define the function on A by $g(\xi) = \text{cf}(\xi)$. Since

$$g(\xi) = \text{cf}(\xi) \leq \kappa < \xi \qquad \text{for all } \xi \in A$$

g is regressive on A. Since A is stationary, there is a stationary $B \subseteq A$ and cardinal $\lambda \leq \kappa$ such that

$$g(\xi) = \text{cf}(\xi) = \lambda \qquad \text{for all } \xi \in B.$$

Proof – continued

- For each $\xi \in B$, choose a strictly increasing cofinal sequence in ξ:

$$\langle \nu_\xi(\eta) \,|\, \eta < \lambda \rangle$$

so that $\sup_{\eta<\lambda} \nu_\xi(\eta) = \xi$.

- For each $\eta < \lambda$ define f_η on B by

$$f_\eta(\xi) \;=\; \nu_\xi(\eta) \qquad \text{for each } \xi \in B.$$

So, for each $\eta < \lambda$, f_η is regressive on B.

- We will show that there is an $\eta < \lambda$ for which the function f_η is not almost bounded on B. By the previous lemma, this is sufficient for partitioning B (and hence A) into κ^+ many pairwise disjoint stationary sets.

Proof – continued

- Suppose that f_η is almost bounded on B for each $\eta < \lambda$.
  This means that for each $\eta < \lambda$, there is a bound $\rho_\eta < \kappa^+$ with

$$B_\eta = \{\xi \in B \,|\, f_\eta(\xi) \geq \rho_\eta\} \;\in\; \mathtt{NS}(\kappa^+).$$

Since $\mathtt{NS}(\kappa^+)$ is κ^+-complete, the following set is stationary

$$S = B - \bigcup_{\eta<\lambda} B_\eta.$$

Define $\rho = \sup_{\eta<\lambda} \rho_\eta < \kappa^+$ and let $\alpha \in S$ with $\rho < \alpha$.

- Now we have our contradiction: by the definition of B_η

$$\nu_\alpha(\eta) \;=\; f_\eta(\alpha) \;<\; \rho_\eta$$

so that

$$\alpha \;=\; \sup_{\eta<\lambda} \nu_\eta(\alpha) \;\leq\; \sup_{\eta<\lambda} \rho_\eta \;=\; \rho \qquad ↯$$

Stationary sets

☞ Stationary sets and regressive functions were first introduced by Bloch (1953). However, the most significant early results (including the Pushdown Theorem) were due to Fodor (1956).

☞ That a stationary set can be partitioned into a number of disjoint stationary sets was proven for $\kappa = \omega_1$ by the Russian topologists Alexandroff and Urysohn (1929). They were using their work on the order topology of ω_1 to provide counterexamples to familiar topological properties of the real line.

☞ Solovay's partition theorem, for successor cardinals was first established by Fodor in 1966. Solovay provided the more difficult regular limit cardinal case in 1971.