Math 582
Intro to Set Theory
Lecture 34

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Introduction

☞ This lecture is an introduction to the club filter and its closure properties. It corresponds to the material from H+J sections 11.1, 11.2 and the first half of section 11.3. See also the first half of Lectures 29 on using Tukey's Lemma to produce ultrafilters.

Definition: Filter and Ideal

Definition

Let $A \neq \emptyset$.

☞ (Cartan, 1937) A filter on A is a nonempty family $\mathcal{F} \subseteq \mathcal{P}(A)$ satisfying

(a) $\emptyset \notin \mathcal{F}$,

(b) If $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$,

(c) If $X \in \mathcal{F}$ and $X \subseteq Y$, then $X \in \mathcal{F}$.

☞ (Stone, 1934) An ideal on A is a nonempty family $\mathcal{I} \subseteq \mathcal{P}(A)$ satisfying

(a') $A \notin \mathcal{I}$,

(b') If $X, Y \in \mathcal{I}$, then $X \cup Y \in \mathcal{I}$,

(c') If $X \in \mathcal{I}$ and $X \supseteq Y$, then $Y \in \mathcal{I}$.

Filters and Ideals

Filters and Ideals as dual concepts

☞ The concept of ideal is the dual concept to that of filter:

- Replace $\emptyset, \cap, \subseteq$ in (a-c) of the filter definition with $A, \cup, \supseteq$ in (a'-c') of ideal definition.

☞ If $\mathcal{F} \subseteq \mathcal{P}(A)$ is a filter on A, then its dual ideal is obtained by taking complements:

$$\mathcal{I} = \{A - X \mid X \in \mathcal{F}\}.$$

and dually, starting with an ideal we can form its dual filter.

Example. For κ a cardinal,

$$\mathcal{I}_\kappa = \{X \subseteq \kappa \mid |X| < \kappa\}.$$

is an ideal on κ whose dual filter is

$$\mathcal{F}_\kappa = \{X \subseteq \kappa \mid |A - X| < \kappa\}$$

Filters and Ideals as dual concepts

Definition

☞ A filter $\mathcal{F}$ on A is κ-complete if $\mathcal{F}$ is closed under $< \kappa$ intersections of its members:

$$G \subseteq \mathcal{F} \ \wedge \ |G| < \kappa \ \rightarrow \ \bigcap G \in \mathcal{F}.$$

☞ An ideal $\mathcal{I}$ on A is κ-complete if $\mathcal{I}$ is closed under $< \kappa$ unions of its members:

$$J \subseteq \mathcal{I} \ \wedge \ |J| < \kappa \ \rightarrow \ \bigcup J \in \mathcal{I}.$$

Example. The ideal $\mathcal{I}_\kappa$ on κ

$$\mathcal{I}_\kappa = \{X \subseteq \kappa \,\big|\, |X| < \kappa\}.$$

is κ-complete, but not κ^+-complete.

Members of an ideal as Insignificant sets

☞ An ideal $\mathcal{I}$ over a set A is often chosen because the members of the ideal are regarded as the “infinitely small" or “insignificant" subsets of A.

For example, the subsets of κ of “small cardinality".

(a') A itself is NOT insignificant,

(b') If X and Y is insignificant, then so is their union $X \cup Y$,

(c') If X is insignificant and $X \supseteq Y$, then Y is also insignificant.

☞ The dual filter $\mathcal{F}$ over a set A can be regarded as the subsets of A which contain “nearly all of A", except for an insignificant part.

Definition: Club sets

Convention. All cardinals κ that we consider in this lecture will be regular and uncountable – unless we explicitly state otherwise.

Definition

- ☞ A subset $C \subseteq \kappa$ is closed if and only if $\sup C \cap \alpha \in C$ for all $\alpha < \kappa$.
- ☞ A subset $C \subseteq \kappa$ is unbounded in κ if $\sup C = \kappa$.
- ☞ A subset $C \subseteq \kappa$ is a club in κ if C is closed and unbounded in κ.

Finite Intersection Property

☞ Clubs have the finite intersection property.

Lemma

If A and B are clubs in κ, then $A \cap B$ is also a club in κ.

☞ It is essential that $\mathrm{cf}(\kappa) > \omega$. For example, the following are clubs in $\aleph_\omega$ with empty intersection:

$$A = \{\aleph_{2n} \mid n \in \omega\} \qquad B = \{\aleph_{2n+1} \mid n \in \omega\}$$

Proof

☞ The intersection of two closed sets is always closed.

☞ It remains to show that $A \cap B$ is unbounded.

Let $\alpha \in \kappa$. We define an increasing sequence $\langle c_n \mid n \in \omega \rangle$ by recursion, using the fact that A and B are unbounded.

$$\begin{aligned} c_0 &= \text{least in } A \text{ greater than } \alpha. \\ c_{2n} &= \text{least in } A \text{ greater than } c_{2n-1}. \\ c_{2n+1} &= \text{least in } B \text{ greater than } c_{2n}. \end{aligned}$$

Let $\gamma = \sup_k c_k$. Since A and B are closed,

$$\begin{aligned} \sup_n c_{2n} &= \gamma \in A \\ \sup_n c_{2n+1} &= \gamma \in B \end{aligned}$$

Thus, $\alpha < \gamma \in A \cap B$.

The club filter

☞ Since the family of clubs in κ has the finite intersection property, they generate a filter on κ, the club filter,

$$\mathcal{F}_\kappa^\clubsuit = \{X \subseteq \kappa \mid X \supseteq C \text{ for some club } C \supseteq \kappa\}$$

☞ $\mathcal{F}_\kappa^\clubsuit \neq \emptyset$ since κ itself is a club. Slightly less trivially

- The set of limit points of κ is a club. Furthermore, if C is any club, the limit points within C is a club.
- More generally, if $C \subseteq \kappa$ is an unbounded set, then the set of limit points of C is a club.

  (We say α is a limit point of a set of ordinals C if for every $\gamma < \alpha$ there is a $\beta \in C$ with $\gamma < \beta < \alpha$.)

Stronger Closure Property of Club Filter

☞ The club filter satisfies a much stronger closure property then the finite intersection property: it is κ-complete.

Lemma

For each $\lambda < \kappa$ and each family of clubs of κ $\{C_\xi \mid \xi < \lambda\}$, the intersection $\bigcap_{\xi<\lambda} C_\xi$ is also a club.

Proof

☞ The intersection of any number of closed sets is always closed.

☞ It remains to show that $\bigcap_{\xi<\lambda} C_\xi$ is unbounded.
Let $\alpha \in \kappa$. Construct a sequence $\langle c_\delta \mid \delta < \lambda \cdot \omega\rangle$ by transfinite induction. To simplify notation recall that the type of $\omega \times \lambda$ under lexicographic order is $\lambda \cdot \omega$.
By recursion on $\delta = (n, \xi)$ with $\xi < \lambda,\ n < \omega$ (and using C_ξ unbounded)

$$c_{\delta=(n,\xi)} = \text{least in } C_\xi \text{ greater than } \alpha \text{ and } c_{\delta'=(n',\xi')} \text{ with } \delta' < \delta.$$

Let $\gamma = \sup_\delta c_\delta$. Since each C_ξ is closed and κ is regular

$$\sup_{n\in\omega} c_{\delta=(n,\xi)} = \gamma \in C_\xi$$

Thus, $\alpha < \gamma \in \bigcap_{\xi<\lambda} C_\xi$.

Normal functions defined

☞ Recall the definition of a normal function from Lecture 20. We now restrict the definition to a regular uncountable cardinal κ.

Definition

Consider a function $f : \kappa \to \kappa$.

- f is order preserving if $\forall \alpha, \beta \in \kappa \left(\alpha < \beta \ \to \ f(\alpha) < f(\beta)\right)$.
- f is continuous if for every limit ordinal $\alpha < \kappa$,
  $f(\alpha) = \sup\{f(\beta) \mid \beta < \alpha\}$.
- f is normal if f is order preserving and continuous.

☞ All results about normal functions in Lecture 20 carry-over to normal functions on κ. Note that the fixed-point construction in slide 16 does require that κ have uncountable cofinality, since the fixed-point constructed has countable cofinality.

Normal functions

☞ The next result provides an alternative characterization of clubs as the range of normal functions.

Theorem

A set $C \subseteq \kappa$ is a club in κ if and only if it is the range of a normal function $f : \kappa \to \kappa$.

An immediate corollary is

Corollary

Let $f : \kappa \to \kappa$ be a normal function. Then the set of fixed-points

$$C = \{\alpha < \kappa \mid f(\alpha) = \alpha\}$$

is a club in κ.

Proof

☞ If $f : \kappa \to \kappa$ is a normal function, then $C = \text{ran}(f)$ is clearly a club in κ.

☞ Let $C \subseteq \kappa$ is a club. Define a function $f : \kappa \to \kappa$ by transfinite recursion

$$f(\alpha) = \text{ least member of } C - f[\alpha] \; .$$

Since C is unbounded in κ, which is regular, $|C| = \kappa$, so that f is well-defined on κ. Furthermore, f is clearly order-preserving by its definition.

☞ Let $\alpha < \kappa$ be a limit ordinal. Since $f[\alpha] \subseteq C$ and C is closed, $\sup f[\alpha] \in C$. Thus, $f(\alpha) = \sup f[\alpha] = \sup_{\xi<\alpha} f(\xi)$. Hence f is continuous.

Therefore, $f : \kappa \to \kappa$ is normal.

More ♣s

☞ The next result generalizes the previous result that the fixed points of normal functions on κ are clubs.

Definition. We say α is a closure point of a function $f : \kappa \to \kappa$ if $f[\alpha] \subseteq \alpha$ (that is, $f(\xi) < \alpha$ whenever $\xi < \alpha$.)

Lemma

Let $f : \kappa \to \kappa$. Then the set of closure points of f,

$$C = \{\alpha < \kappa \,|\, f[\alpha] \subseteq \alpha\},$$

is a club.

Proof

Let

$$C = \{\alpha < \kappa \mid f[\alpha] \subseteq \alpha\},$$

☞ Since $f[\alpha] = \bigcup_{\beta<\alpha} f[\beta]$ for any limit ordinal α, C is closed.

☞ It remains to show C is unbounded. Fix $\alpha_0 < \kappa$. Define by recursion an increasing sequence $\langle \alpha_n \mid n \in \omega \rangle$ by

$$\alpha_{n+1} \quad = \quad \text{least in } \kappa \text{ greater than } \alpha_n \text{ with } f[\alpha_n] \subseteq \alpha_{n+1}$$

Since κ is regular, $f[\delta]$ is bounded for each $\delta < \kappa$, so the sequence is well defined. Let

$$\alpha = \sup_n \alpha_n.$$

Then

$$f[\alpha] \;=\; \bigcup_n f[\alpha_n] \;\subseteq\; \bigcup_{n+1} f[\alpha_{n+1}] \;=\; \alpha$$

so that $\alpha \in C$.

Diagonal intersection

☞ It is certainly not true that clubs on κ are closed under κ-intersections: let $C_\alpha = \kappa - \alpha$, and so

$$\bigcap_{\alpha<\kappa} C_\alpha = \emptyset.$$

However, the next lemma provides an extension to κ many clubs.

Lemma

For any κ-sequence of clubs in κ, $\langle C_\alpha \mid \alpha < \kappa \rangle$, the diagonal intersection defined by

$$\Delta_{\alpha<\kappa} C_\alpha = \{\gamma \in \kappa \mid \forall \xi < \gamma \,[\gamma \in C_\xi]\}$$

is a club in κ.

Proof

$$C = \Delta_{\alpha<\kappa} C_\alpha = \{\gamma \in \kappa \,|\, \forall \xi < \gamma \, [\gamma \in C_\xi]\}$$

- Show C is closed. Let γ be a limit ordinal. We may assume $C \cap \gamma$ is unbounded below γ, otherwise replace γ by $C \cap \gamma$ and argue as below.
- There is a strictly increasing sequence $\langle \gamma_\xi \,|\, \xi < \mathrm{cf}(\gamma)\rangle$ from C with $\gamma = \sup_{\xi<\mathrm{cf}(\gamma)} \gamma_\xi$. Note, by definition, for each $\alpha < \mathrm{cf}(\gamma)$:

$$\gamma_\xi \in \bigcap_{\beta<\gamma_\alpha} C_\beta \qquad \text{for each } \xi > \alpha,$$

so that $\gamma = \sup_{\xi>\alpha} \gamma_\xi \in \bigcap_{\beta<\gamma_\alpha} C_\beta$, as this set is closed.
- Thus,

$$\gamma \in \bigcap_{\alpha<\mathrm{cf}(\gamma)} \Big(\bigcap_{\beta<\gamma_\alpha} C_\beta \Big) = \bigcap_{\delta<\gamma} C_\delta;$$

and so $\gamma \in C$.

Proof – continued

$$C = \Delta_{\alpha<\kappa} C_\alpha = \{\gamma \in \kappa \,|\, \forall \xi < \gamma \, [\gamma \in C_\xi]\}$$

- Show C is unbounded. Note that any $\gamma < \kappa$, $\bigcap_{\xi<\gamma} C_\gamma$ is a club.
- Fix $\alpha_0 \in \kappa$. Define a sequence $\langle \alpha_n \,|\, n < \omega\rangle$ by recursion:

$$\alpha_{n+1} = \text{least ordinal in } \bigcap_{\xi<\alpha_n} C_\xi.$$

Let $\alpha = \sup_n \alpha_n$. Then for each n,

$$\{\alpha_{n+1}, \alpha_{n+2}, \dots\} \subseteq \bigcap_{\xi<\alpha_n} C_\xi.$$

so that $\alpha \in \bigcap_{\xi<\alpha_n} C_\xi$, as this set is closed. Thus,

$$\alpha \in \bigcap_{n<\omega} \Big(\bigcap_{\xi<\alpha_n} C_\xi \Big) = \bigcap_{\delta<\alpha} C_\delta;$$

and so $\alpha \in C$.