Math 582
Intro to Set Theory
Lecture 33

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Cardinal exponentiation

☞ There are few results that say something significant about the behavior of the value κ^λ. These are the most important:

(A) $\kappa^\lambda = 2^\lambda$ when $2 \leq \kappa \leq 2^\lambda$,

(B) $\kappa^{\mathrm{cf}(\kappa)} > \kappa$ (for $\kappa \geq \omega$),

(C) When $\kappa \geq \omega$ and $0 < \lambda < \mathrm{cf}(\kappa)$ we have

$$\kappa^\lambda = \Big(\sum_{\tau<\kappa} \tau^\lambda\Big) \cdot \kappa$$

where τ runs over cardinals.

☞ We will see that these three results are sufficient for determining the value of κ^λ under GCH.

Note. For (A) see König's Lemma on Lecture 32, slide 4, and Lemma 9.3.9.

König's Theorem revisited (B)

☞ The crux of cardinal exponentiation is determining $\kappa^{\mathrm{cf}(\kappa)}$.
The following is about all we can say about this value:

Theorem

For each infinite cardinal κ,

$$\kappa^{\mathrm{cf}(\kappa)} > \kappa.$$

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König's Theorem revisited (B)

Proof.

☞ If κ is regular, then by (A):

$$\kappa^{\mathrm{cf}(\kappa)} = \kappa^{\kappa} = 2^{\kappa}$$

☞ Suppose κ is singular. Fix an increasing family of cardinals $\langle \kappa_\xi \mid \xi < \mathrm{cf}(\kappa) \rangle$ with $0 < \kappa_\xi < \kappa$ for all ξ and $\kappa = \sum_{\xi < \mathrm{cf}(\kappa)} \kappa_\xi$.

Since $\kappa_\xi < \kappa_{\xi+1}$ we use König's Theorem:

$$\kappa = \sum_{\xi < \mathrm{cf}(\kappa)} \kappa_\xi < \prod_{\xi < \mathrm{cf}(\kappa)} \kappa_{\xi+1} \leq \prod_{\xi < \mathrm{cf}(\kappa)} \kappa = \kappa^{\mathrm{cf}(\kappa)}$$

□

König's Theorem revisited (B)

☞ The following generalizes our $\mathrm{cf}(2^\kappa) > \kappa$.
See Lecture 32, slide 6 and H+J, Lemma 3.3. See H+J, Lemma 9.3.7 for a statement and slightly different proof.

Corollary

For each cardinal $\kappa > 1$ and infinite cardinal λ, $\mathrm{cf}(\kappa^\lambda) > \lambda$.

Proof.

By the previous theorem,

$$(\kappa^\lambda)^{\mathrm{cf}(\kappa^\lambda)} > \kappa^\lambda.$$

If $\tau \leq \lambda$, then

$$(\kappa^\lambda)^\tau = \kappa^{\lambda\cdot\tau} = \kappa^\lambda.$$

Therefore, we must have $\mathrm{cf}(\kappa^\lambda) > \lambda$. □

Bernstein-Hausdorff-Tarski Theorem

☞ The next theorem says that κ^λ can be calculated from τ^λ for cardinals $\tau < \kappa$, when λ is "small" relative to κ – that is, $\lambda < \mathrm{cf}(\kappa)$.

Theorem

Let $\kappa \geq \omega$ and $0 < \lambda < \mathrm{cf}(\kappa)$ be cardinals. Then, with τ running over cardinals,

$$\kappa^\lambda = \Big(\sum_{\tau<\kappa} \tau^\lambda\Big) \cdot \kappa$$

Note. H+J's Hausdorff's Formula (9.3.11) is really a special case of this result for κ a successor cardinal. In the case, where $\kappa = \aleph_{\alpha+1}$, if $\lambda < \mathrm{cf}(\aleph_{\alpha+1})$ we get exactly the special case here: $\aleph_{\alpha+1}^\lambda = \aleph_\alpha^\lambda \cdot \aleph_{\alpha+1}$; if $\lambda \geq \mathrm{cf}(\aleph_{\alpha+1}) = \aleph_{\alpha+1}$ we get this result by appealing to condition (A): $\aleph_{\alpha+1}^\lambda = 2^\lambda = \aleph_\alpha^\lambda$. The theorem stated in this slide also covers the case for limit cardinals κ as well.

Proof of Bernstein-Hausdorff-Tarski

Let $\kappa \geq \omega$ and $0 < \lambda < \text{cf}(\kappa)$ be cardinals.

☞ $\kappa \leq \kappa^\lambda$ (since $\lambda > 0$) and $\tau^\lambda \leq \kappa^\lambda$ (since cardinal exponentiation is monotonic.) Thus,

$$\kappa^\lambda \geq \Big(\sum_{\tau<\kappa} \tau^\lambda\Big) \cdot \kappa$$

By H+J Theorem 9.1.3 or Lecture 30, Slide 16.

☞ Since $\kappa^\lambda = \left|{}^\lambda\kappa\right|$, we show the converse by establishing

$${}^\lambda\kappa = \bigcup_{\xi<\kappa} {}^\lambda\xi,$$

where ξ ranges over ordinals.

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Proof of Bernstein-Hausdorff-Tarski

☞ Suppose $f \in {}^\lambda\kappa$. Since $\lambda < \text{cf}(\kappa)$ we must have ran(f) is bounded in κ, say ran$(f) \subseteq \xi$. Thus, we have established $\subseteq$ of

$${}^\lambda\kappa = \bigcup_{\xi<\kappa} {}^\lambda\xi$$

(the converse is clear.)

☞ Now, (where $\tau < \kappa$ ranges over cardinals and $\xi < \kappa$ ranges over ordinals)

$$\kappa^\lambda = \sum_{\xi<\kappa} \left|\xi^\lambda\right| = \sum_{\xi<\kappa} |\xi|^\lambda \leq \sum_{\tau<\kappa} (\tau^\lambda \cdot \kappa) = \Big(\sum_{\tau<\kappa} \tau^\lambda\Big) \cdot \kappa$$

The third inequality is because $\kappa > \left|\{\xi < \kappa \,\middle|\, |\xi| = \tau\}\right|$ for any $\tau < \kappa$.

Proof of Bernstein-Hausdorff-Tarski

Bernstein's Theorem. Bernstein originally proved the following

$$\aleph_n^{\aleph_0} = 2^{\aleph_0} \cdot \aleph_n \quad \text{for all } n < \omega$$

Since $\aleph_0 < \mathrm{cf}(\aleph_n)$ we can apply the Bernstein-Hausdorff-Tarski Theorem. The proof is by induction on n.

☞ $(n = 0)$. $\aleph_0^{\aleph_0} = 2^{\aleph_0} = 2^{\aleph_0} \cdot \aleph_0$ by (A).

Assume the assertion is true for n. Then, by the previous theorem

$$\begin{aligned} \aleph_{n+1}^{\aleph_0} &= \Big(\sum_{i \le n} \aleph_i^{\aleph_0} + \sum_{k < \omega} k^{\aleph_0}\Big) \cdot \aleph_{n+1} \\ &= \Big(\sum_{i \le n} \aleph_i^{\aleph_0} + 2^{\aleph_0} \cdot \aleph_0\Big) \cdot \aleph_{n+1} \\ &= \aleph_n^{\aleph_0} \cdot \aleph_{n+1} \\ &= (2^{\aleph_0} \cdot \aleph_n) \cdot \aleph_{n+1} \qquad \text{i.h.} \\ &= 2^{\aleph_0} \cdot \aleph_{n+1} \end{aligned}$$

Cardinal exponentiation with GCH

☞ GCH is the statement that $2^\kappa = \kappa^+$ for all κ.

Theorem

Assume GCH. Let κ be an infinite cardinal and $\lambda > 0$, then

$$\kappa^\lambda = \begin{cases} \kappa & \text{if } \lambda < cf(\kappa) \\ \kappa^+ & \text{if } cf(\kappa) \le \lambda \le \kappa \\ \lambda^+ & \text{if } \lambda \ge \kappa \end{cases}$$

Note. Hrbacek and Jech break this into two cases: Theorem 3.8 (regular) and Theorem 3.10 (singular).

Cardinal exponentiation with GCH

☞ Suppose $\lambda < \text{cf}(\kappa)$. By the Bernstein-Hausdorff-Tarksi Theorem

$$\begin{aligned} \kappa &\leq \kappa^\lambda \\ &= \Big(\sum_{\tau<\kappa} \tau^\lambda\Big) \cdot \kappa \\ &\leq \kappa\Big(\sum_{\tau<\kappa} 2^{\tau\cdot\lambda}\Big) \\ &\leq \kappa\Big(\sum_{\tau<\kappa} \max\{\tau^+, \lambda^+\}\Big) \\ &\leq \kappa \cdot \kappa = \kappa. \end{aligned}$$

So, $\kappa = \kappa^\lambda$.

Cardinal exponentiation with GCH

☞ Suppose $\text{cf}(\kappa) \leq \lambda \leq \kappa$. Then,

$$\kappa < \kappa^{\text{cf}(\kappa)} \leq \kappa^\lambda \leq \kappa^\kappa \leq 2^\kappa = \kappa^+.$$

(The first '$<$' is by (B), the second '$\leq$' is by (A).)

Therefore, $\kappa < \kappa^\lambda \leq \kappa^+$, so $\kappa^\lambda = \kappa^+$.

☞ Suppose $\lambda \geq \kappa$. Then $\kappa^\lambda = 2^\lambda = \lambda^+$ by (A) .

Exponentiation and the power function

☞ GCH completely determines the power function, 2^κ for all κ, and this is sufficient to compute κ^λ for all cardinals κ and λ.

☞ Unfortunately, it is possible to fix the value of the power function 2^κ for all κ, and still not be able to compute in ZFC κ^λ for all cardinals κ and λ.

☞ ZFC is consistent with the statement:

$$(\maltese)\quad 2^{\aleph_0} = \aleph_1,\ 2^{\aleph_n} = \aleph_{\omega+2}\ (\text{for all } 1 < n \leq \omega),\ 2^\kappa = \kappa^+\ (\kappa > \aleph_\omega),$$

together with either of the possibilities that (i) $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$ or (ii) $\aleph_\omega^{\aleph_0} = \aleph_{\omega+2}$.

☞ Thus, ZFC + ($\maltese$) completely determines the power function, but leaves open $\aleph_\omega^{\aleph_0}$. It turns-out that this is the only kind of gap that needs to be filled.

Gimel Function

Definition

The gimel function, $\gimel$, is the function on the infinite cardinals defined by

$$\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}$$

- ✎ If κ is regular then $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)} = \kappa^\kappa = 2^\kappa$. (by (A).)
- ✎ If κ is singular then $\kappa < \gimel(\kappa) \leq 2^\kappa$ (by (B).)

Theorem

The gimel function completely determines the power function $\kappa \mapsto 2^\kappa$ and cardinal exponentiation $(\kappa, \lambda) \mapsto \kappa^\lambda$.

Proof: Power function

$\gimel$ determines $\kappa \mapsto 2^\kappa$. The proof is by transfinite induction on κ.

☞ If κ is regular, then $2^\kappa = \kappa^\kappa = \kappa^{\text{cf}(\kappa)} = \gimel(\kappa)$ (by (A).)

☞ Suppose κ is singular, and suppose that $\tau \mapsto 2^\tau$ has been determined for all $\tau < \kappa$.

Fix an increasing sequence of cardinals with $\kappa = \sum_{\xi<\text{cf}(\kappa)} \kappa_\xi$ where $\kappa_\xi < \kappa$ for all $\xi < \text{cf}(\kappa)$. Two cases.

Case (i). 2^τ is eventually constant. Then $2^\kappa = 2^\tau$ where 2^τ is this constant value (Lecture 32, slide 9.) Since $\tau < \kappa$, the value of 2^κ is determined.

Proof: Power function

Case (ii). 2^τ is not eventually constant. Let $\lambda = \sup_{\xi<\text{cf}(\kappa)} 2^{\kappa_\xi}$.

Then $\text{cf}(\lambda) = \text{cf}(\kappa)$ (① of Lecture 23, slide 7) and $\lambda \leq 2^\kappa$ (since $2^\kappa \geq 2^{\kappa_\xi}$ for each ξ.)

Compute:

$$\begin{aligned} 2^\kappa &= 2^{\sum_{\xi<\text{cf}(\kappa)} \kappa_\xi} \\ &= \prod_{\xi<\text{cf}(\kappa)} 2^{\kappa_\xi} && \text{Homework 12, Problem 1} \\ &\leq \lambda^{\text{cf}(\kappa)} = \lambda^{\text{cf}(\lambda)} \\ &\leq \left(2^\kappa\right)^{\text{cf}(\lambda)} \\ &\leq 2^\kappa. \end{aligned}$$

So, $2^\kappa = \lambda^{\text{cf}(\lambda)} = \gimel(\lambda)$.

Proof: Cardinal exponentiation

The $\gimel$ determines $(\kappa, \lambda) \mapsto \kappa^\lambda$. We assume the power function $\kappa \mapsto 2^\kappa$ is determined. The proof is by transfinite induction on κ.

☞ If $0 < \lambda < \omega$, then $\kappa^\lambda = \kappa$.

☞ If $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \left(\sum_{\tau<\kappa} \tau^\lambda\right) \cdot \kappa$ by Bernstein-Hausdorff-Tarski. Each τ^λ for $\tau < \lambda$ is determined by the inductive hypothesis.

☞ If $\lambda = \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \gimel(\kappa)$.

☞ If $\lambda \geq \kappa$, then $\kappa^\lambda = 2^\lambda$ (by (A)), which is determined since the power function is determined.

Cardinal Exponentiation without GCH

Proof: Cardinal exponentiation

☞ Suppose $\mathrm{cf}(\kappa) < \lambda < \kappa$. This case implies κ is singular.

Case (i). There is a $\tau < \kappa$ with $\tau^\lambda > \kappa$. Then

$$\tau^\lambda \leq \kappa^\lambda \leq (\tau^\lambda)^\lambda = \tau^\lambda.$$

So, $\kappa^\lambda = \tau^\lambda$, which determined by the inductive hypothesis.

Case (ii). $\tau^\lambda < \kappa$ for each $\tau < \kappa$.

Let $\langle \kappa_\xi \,\big|\, \xi < \mathrm{cf}(\kappa) \rangle$ be increasing with $\kappa = \sum_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi$.

$$\begin{aligned}
\kappa^\lambda &= \Big(\sum_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi\Big)^\lambda \\
&\leq \Big(\prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi\Big)^\lambda && \text{Homework 12, Problem 2} \\
&= \prod_{\xi<\mathrm{cf}(\kappa)} \kappa_\xi^\lambda && \text{Homework 12, Problem 1} \\
&\leq \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\lambda
\end{aligned}$$

So, $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)} = \gimel(\kappa)$.