Math 582
Intro to Set Theory
Lecture 32

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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The cardinal power function

☞ Determining the values of the cardinal power function, $\kappa \mapsto 2^\kappa$, is of fundamental concern in set theory, from its earliest history with the continuum hypothesis of Cantor: $2^{\aleph_0} = \aleph_1$.

☞ Felix Hausdorff soon generalized this statement to the generalized continuum hypothesis: $2^\kappa = \kappa^+$ for every κ. Both Cantor's continuum hypothesis and Hausdorff's generalized continuum hypothesis were shown to be independent of ZFC (by Gödel in 1938 and Cohen in 1963.)

☞ We will look into what can be proven in ZFC (not much), and how flexible we can be in defining the power function in ZFC (maximally flexible). This leads us to one of the fundamental problems in set theory today, the singular cardinal problem. We will explore a little to the limits of what is known.

The cardinal power function

☞ There are two general facts about cardinal power function $\kappa \mapsto 2^\kappa$: for all cardinals κ and λ,

$$\begin{array}{ll} (\mathbf{A}) & \kappa < \lambda \longrightarrow 2^\kappa \leq 2^\lambda \\ (\mathbf{B}) & \mathrm{cf}(2^\kappa) > \kappa \end{array}$$

☞ We cannot replace '$\leq$' by '$<$' in (**A**). For example, it is consistent with ZFC that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

☞ Cantor's theorem is a consequence of (**B**): $2^\kappa > \kappa$ for every cardinal κ.

☞ (**A**) and (**B**) are the only rules in ZFC which govern the behavior of the power function $\kappa \mapsto 2^\kappa$, for regular cardinals.

König's Lemma

(**B**): A consequence of König's Theorem:

Lemma (König's Lemma)

For every infinite cardinal κ, cf$(2^\kappa) > \kappa$.

More generally, for every $\mu > 1$, cf$(\mu^\kappa) > \kappa$ (H+J, Lemma 3.3.9.)

☞ König's Lemma rules-out the following possibilities for $\mathrm{cf}(2^{\aleph_0})$

- $2^{\aleph_0} = \aleph_0$ (Cantor's theorem is a conseqence),
- $2^{\aleph_0} = \aleph_\omega$,
- $2^{\aleph_0} = \aleph_{\aleph_\omega}$.

Proof of König's Lemma

Proof.

We give the proof for $\mu = 2$, although the general case is obtained by replacing 2 by μ everywhere. (We will prove the more general case in a different way in the next lecture.)

☞ Let $\zeta = \mathrm{cf}(2^\kappa)$. So, there is a family of cardinals $\langle \lambda_\nu \mid \nu < \zeta \rangle$ with $\lambda_\nu < 2^\kappa$ for all $\nu < \zeta$, and

$$\sum_{\nu<\zeta} \lambda_\nu = 2^\kappa$$

By König's Theorem:

$$2^\kappa = \sum_{\nu<\zeta} \lambda_\nu < \prod_{\nu<\zeta} 2^\kappa = (2^\kappa)^\zeta$$

So, if $\zeta \leq \kappa$ we would have

$$2^\kappa < (2^\kappa)^\zeta \leq (2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} = 2^\kappa \quad \lightning.$$

□

Continuum Hypothesis

☞ Two general facts about cardinal power function $\kappa \mapsto 2^\kappa$: for all cardinals κ and λ,

$$\textbf{(A)} \qquad \kappa < \lambda \longrightarrow 2^\kappa \leq 2^\lambda$$
$$\textbf{(B)} \qquad \mathrm{cf}(2^\kappa) > \kappa$$

☞ There is nothing more than (**B**) to constrain what $2^{\aleph_0}$ could be:

- ➢ $2^{\aleph_0} = \aleph_1$ (the Continuum Hypothesis)
- ➢ $2^{\aleph_0} = \aleph_2$ (Gödel, Woodin and some current set theorists argue for this),
- ➢ $2^{\aleph_0} = \aleph_{27}$ (my favorite choice – why not?)
- ➢ $2^{\aleph_0} = \aleph_{\omega_1}$ (singular cardinals are possible)

☞ It is possible that $2^{\aleph_0}$ to be regular or singular, $\aleph_1$ (as small as possible), or weakly inaccessible (which is very, very large.)

Singular case

There are some restraints on $2^{\aleph_\alpha}$ when $\aleph_\alpha$ is a singular cardinal.

☞ If λ is singular and 2^κ has a fixed value for all $\kappa < \lambda$ from some point on, then 2^λ has that same value.

Theorem (H+J, 9.3.5)

Let λ be a singular cardinal.
If there is a $\mu < \lambda$ such that $2^\kappa = 2^\mu$ for every κ with $\mu < \kappa < \lambda$, then $2^\lambda = 2^\mu$.

Proof of 9.3.5

☞ Let λ be a singular cardinal and suppose there is a $\mu < \lambda$ such that $2^\kappa = 2^\mu$ for all κ with $\mu < \kappa < \lambda$.

☞ There is a cardinal $\zeta = \mathrm{cf}(\lambda) < \lambda$ and an increasing family of cardinals $\langle \kappa_\nu \mid \nu < \zeta \rangle$ with $\sum_{\nu<\zeta} \kappa_\nu = \lambda$ and $\kappa_\nu \geq \mu$ for each ν; so,

$$\begin{aligned} 2^\lambda &= 2^{\sum_{\nu<\zeta} \kappa_\nu} \\ &= \prod_{\nu<\zeta} 2^{\kappa_\nu} && \text{Homework 12 ☺} \\ &\leq \prod_{\nu<\zeta} 2^\mu \\ &= 2^{\mu\cdot\zeta} \\ &= 2^\mu && \text{since } \mu \leq \mu\cdot\zeta < \lambda \end{aligned}$$

Generalized Continuum Hypothesis

☞ The Generalized Continuum Hypothesis (GCH) is the statement $\forall\alpha\left[2^{\aleph_\alpha} = \aleph_{\alpha+1}\right]$.

☞ (GCH) completely decides cardinal exponentiation, $\kappa \mapsto \kappa^\lambda$. (Next Lecture.)

☞ In 1938 Kurt Gödel showed that GCH cannot be disproved from the other axioms of ZFC. In 1963 Paul Cohen (using the method of forcing) showed that ZFC cannot prove $2^{\aleph_0} = \aleph_1$; later, using his forcing technique, it was shown that $2^{\aleph_0}$ can be “arbitrarily large" (including weakly inaccessible.)

Strong Limit Cardinal

Definition

A limit cardinal λ is a strong limit cardinal if $\lambda > 2^\tau$ for all $\tau < \lambda$. A strongly inaccessible cardinal is a regular limit cardinal.

☞ $\aleph_0$ is a strongly inaccessible cardinal.

☞ Under GCH every limit cardinal is a strong limit cardinal, and any weakly inaccessible cardinal is strongly inaccessible.

☞ The existence of uncountable strongly inaccessible cardinals cannot be proven (or refuted) in ZFC. It is consistent that the smallest weakly inaccessible cardinal is very much smaller than the smallest strongly inaccessible.

Strong Limit Cardinal

Definition

For ordinals α,

$$\beth_\alpha = \begin{cases} \aleph_0 & \text{if } \alpha = 0 \\ 2^{\beth_\gamma} & \text{if } \alpha = \gamma + 1 \\ \sup\{\beth_\gamma \mid \gamma < \alpha\} & \text{if } \alpha \text{ is a limit.} \end{cases}$$

☞ $\beth$ is a normal function, so has fixed-points: $\alpha = \beth_\alpha$. These fixed-points are strong limits, but not necessarily regular.

☞ The usual way of producing fixed points produces singular cardinals:

Let κ be any infinite cardinal:

$$\beta_0 = \kappa \quad \beta_{n+1} = 2^{\beta_n} \quad \beta = \sup_{n<\omega} \beta_n.$$

Then β is a strong limit, but not regular, since $\text{cf}(\beta) = \omega < \beta$.

Power Function underdetermined

☞ William Easton (1964) showed that (**A**) and (**B**) are the only rules in ZFC which govern the behavior of the power function $\kappa \mapsto 2^\kappa$ on regular cardinals.

☞ Let F be a class function on the regular cardinal numbers which is reasonably defined and obeys

(A) $\kappa \leq \lambda \ \rightarrow \ F(\kappa) \leq F(\lambda)$,

(B) $\text{cf}(F(\kappa)) > \kappa$

Then it is consistent with ZFC that $2^\kappa = F(\kappa)$ for all regular cardinals.

☞ “Reasonably defined" is a vague, but broad requirement. The following are “reasonable definitions" of F:

- $2^{\aleph_\alpha} = \aleph_{\alpha+13}$ for all regular $\aleph_\alpha$.
- $2^{\aleph_0} = \aleph_{243}$, $2^{\aleph_1} = \aleph_{399}$ and $2^{\aleph_\alpha} = \aleph_{(\aleph_\alpha)^+}$ for all regular $\aleph_\alpha > \aleph_1$.
- $2^{\aleph_n} = \aleph_{\omega+n}$ for all $n < \omega$, $2^{\aleph_\alpha} = \aleph_{\alpha^2+13}$ for all regular even $\alpha \geq \omega$ ($\alpha = 2 \cdot \beta$ for some β) and $2^{\aleph_\alpha} = \aleph_{\alpha^2+47}$ for all regular odd $\alpha \geq \omega$.

The singular cardinal problem

☞ Shortly after Easton's result, Robert Solovay raised the Singular Cardinal Problem: the problem of finding a complete set of rules describing the behavior of the power function on singular cardinals.

☞ The lacuna in our understanding centers on two specific problems:

(a) Is it possible for some singular cardinal to be the first failure for GCH:

For all $\tau < \lambda$, $2^\tau = \tau^+$, but $2^\lambda > \lambda^+$?

(b) If λ is a singular strong limit, is it possible to give a bound in ZFC for 2^λ?

The singular cardinal problem (a)

(a). Is it possible for some singular cardinal to be the first failure for GCH:

For all $\tau < \lambda$, $2^\tau = \tau^+$, but $2^\lambda > \lambda^+$?

☞ In 1975 Jack Silver proved that the first failure of GCH cannot occur at a singular cardinal of uncountable cofinality (partially answering (a)):

Theorem (Silver's Theorem, H+J, 11.4.1)

Let λ be a singular cardinal of uncountable cofinality.
If $2^\kappa = \kappa^+$ for all $\kappa < \lambda$ then $2^\lambda = \lambda^+$.

☞ Could GCH first fail at a singular cardinal with countable cofinality?

The singular cardinal hypothesis

☞ Magidor proved in 1977 that if we assume there exists a certain large cardinal (a supercompact cardinal) then it is consistent that with ZFC that we have the first failure of GCH at a singular cardinal:

$$2^{\aleph_n} = \aleph_{n+1} \ (n < \omega) \qquad 2^{\aleph_\omega} = \aleph_{\omega+2}$$

☞ The Singular Cardinal Hypothesis is the statement

(SCH) $2^\kappa = \kappa^+$ at all singular strong limit cardinals.

(This provides a strong answer to (a).)

The singular cardinal problem (b)

(b). If λ is a singular strong limit, is it possible to bound 2^λ in ZFC?

☞ In 1975 Galvin and Hajnal showed that we can answer (b) affirmatively for singular cardinals of uncountable cardinality; Shelah in 1980 extended their result to countable cardinality:

Theorem (Galvin-Hajnal, Shelah)

If $\aleph_\alpha$ is a singular strong limit cardinal then

$$2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$$

Example. If $2^{\aleph_\alpha} < \aleph_{\omega_1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$.

The singular cardinal problem (b):

Shelah's strengthening

☞ The bound of the previous theorem was considerably tightened by Shelah in the late 80s. When κ is a cardinal, we will write κ^{+n} (where $n < \omega$) for the nth successor of κ: let $\kappa = \aleph_\alpha$ then $\kappa^{+n} = \aleph_{\alpha+n}$.

$$(\aleph_5)^{+2} = (\aleph_5)^{++} = \aleph_7 \quad (\aleph_0)^{+27} = \aleph_{27} \quad \aleph_{(\omega_0)^{+4}} = \aleph_{\omega_4}$$

Theorem (Shelah)

If $\aleph_\alpha$ is a singular strong limit cardinal then

$$2^{\aleph_\alpha} < \aleph_{|\alpha|^{+4}}$$

(The index $|\alpha|^{+4}$ replaces $(2^{|\alpha|})^+$ in the previous slide.)

Major open question. If $\aleph_\omega$ is a singular strong limit cardinal is it possible that $2^{\aleph_\omega} > \aleph_{\omega_1}$?? (The theorem guarantees $2^{\aleph_\omega} < \aleph_{\omega_4}$.)