Math 582
Intro to Set Theory
Lecture 30

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Introduction

☞ We introduce cardinal exponentiation, which requires AC. The basic properties can be found in Hrbacek and Jech, Section 5.1.

☞ Cardinal exponentiation is used to describe the Continuum Hypothesis (CH) and the Generalized Continuum Hypothesis (GCH).

☞ Our goal over the next couple of weeks is to explore what can be said about cardinal exponentiation both with and without the assumption of GCH. There is not much.

☞ To explore cardinal exponentiation we will need to introduce two new concepts: (1) infinite sums and products (H+J, Section 9.1) and (2) cofinality (H+J, Section 9.2). In this lecture we discuss (1); the highlight is König's Theorem (H+J, 9.1.7.)

Cardinal exponentiation defined

☞ The Axiom of Choice implies that all sets are well-orderable, so there is a cardinal number for $\left|{}^{B}A\right|$, the set of functions $B \to A$.

Definition

If κ, λ are cardinals, then

$$\kappa^{\lambda} = \left|{}^{\lambda}\kappa\right|.$$

- If $|A| = \kappa$ and $|B| = \lambda$ then $\left|{}^{B}A\right| = \left|{}^{\lambda}\kappa\right|$. So, the definition does not depend upon our choice of cardinal representatives on the right-side of the equality.
- 0 is an oddity (as with finite cardinals): $0^0 = \left|{}^{0}0\right| = \left|\{\emptyset\}\right| = 1$ (since the empty function maps $\emptyset$ to $\emptyset$), while for $\lambda > 0$, $0^{\lambda} = \left|{}^{\lambda}0\right| = |\emptyset| = 0$.

Cardinal Exponentiation

Cardinal exponentiation defined

☞ The Axiom of Choice is needed for any kind of account of cardinal exponentiation by the following result, due to Rubin in 1960.

Theorem

The Axiom of Choice is equivalent to the following statement

- *The power set of every ordinal is well-orderable.*

Properties of cardinal exponentiation

Lemma

Let κ, λ, θ be cardinals. Then

(i) $\left(\kappa^{\lambda}\right)^{\theta} = \kappa^{\lambda\cdot\theta}$.

(ii) $\kappa^{\lambda+\theta} = \kappa^{\lambda} \cdot \kappa^{\theta}$.

(iii) $\left(\kappa \cdot \lambda\right)^{\theta} = \kappa^{\theta} \cdot \lambda^{\theta}$

Proof.

These follow from the corresponding laws of size: for all A, B, C the following hold:

(i) ${}^{C}\left({}^{B}A\right) \approx {}^{C\times B}A$.

(ii) ${}^{B\cup C}A \approx {}^{B}A \times {}^{C}A$ when $B \cap C = \emptyset$.

(iii) ${}^{A}(B \times C) \approx {}^{A}B \times {}^{A}C$

(See Lecture 24, slide 15; or Hrbacek and Jech, Theorem 5.1.7.) □

Cardinal exponentiation and power sets

☞ Cardinal exponentiation is of fundamental importance because of its relation to the Continuum Hypothesis.

- $2^{\kappa} = \left|\mathcal{P}(\kappa)\right|$ for every cardinal κ.
- $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for all ordinals α.

Definition

- The Continuum Hypothesis (CH) is the statement $2^{\aleph_0} = \aleph_1$.
- The Generalized Continuum Hypothesis (GCH) is the statement $\forall\alpha\left[2^{\aleph_\alpha} = \aleph_{\alpha+1}\right]$.

Bases in exponentiation

Knowing 2^λ for infinite λ tells us alot about exponentiation in other bases:

Lemma

If $2 \leq \kappa \leq 2^\lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$.

Proof.

$$2^\lambda \leq \kappa^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda \cdot \lambda} = 2^\lambda$$

□

Computing cardinal exponentiation

☞ The last result shows that infinite cardinal arithmetic is simpler than cardinal arithmetic. But, there are further complexities about κ^λ when $\lambda < \kappa$.

☞ Assuming GCH simplifies calculations, and in fact completely determines cardinal exponentiation, as we will see. Without GCH there is very little we can say about cardinal exponentiation.

☞ We will explore both possibilities, but first we turn to infinite sums and products, then cofinalities.

Infinite sums

The following generalizes the sum of two cardinal numbers, $\kappa_1 + \kappa_2$:

Definition

Let $\langle A_i \mid i \in I \rangle$ be a system of mutually disjoint sets, and let $\kappa_i = |A_i|$. Then the sum of $\langle \kappa_i \mid i \in I \rangle$ is

$$\sum_{i \in I} \kappa_i = \Big|\bigcup_{i \in I} A_i\Big|$$

- Let $I = \{1, 2\}$, we get $\kappa_1 + \kappa_2$.
- We need AC to show this definition "makes sense". Without AC it is possible that we have two systems of mutually disjoint sets $\langle A_n \mid n < \omega \rangle$ and $\langle B_n \mid n < \omega \rangle$ with $|A_n| = 2 = |B_n|$, but $\bigcup_n A_n \not\approx \bigcup_n B_n$.

Crucial Lemma for definition

Lemma

Let $\langle A_i \mid i \in I \rangle$ and $\langle B_i \mid i \in I \rangle$ be systems of mutually disjoint sets such that $|A_i| = |B_i|$ for every $i \in I$.
Then $\big|\bigcup_{i \in I} A_i\big| = \big|\bigcup_{i \in I} B_i\big|$.

Proof.

For each $i \in I$ choose a bijection $f_i : A_i \rightleftarrows B_i$.
Let $f = \bigcup_{i \in I} f_i$, so $f : \bigcup_{i \in I} A_i \rightleftarrows \bigcup_{i \in I} B_i$.

This choice requires AC. □

Order Lemma

Lemma

If $\kappa_i \leq \lambda_i$ for all $i \in I$ then

$$\sum_{i \in I} \kappa_i \leq \sum_{i \in I} \lambda_i.$$

Note. It does not generally hold that if $\kappa_i < \lambda_i$ for all $i \in I$ then

$$\sum_{i \in I} \kappa_i < \sum_{i \in I} \lambda_i.$$

For example, $1 < \aleph_0$ but

$$\sum_{n<\omega} 1 = \aleph_0 = \sum_{n<\omega} \aleph_0.$$

since $\sum_{n<\omega} 1 = \aleph_0 = \aleph_0 \cdot \aleph_0 = \sum_{n<\omega} \aleph_0$.

Proof of Order Lemma

Proof.

Let $\kappa_i \leq \lambda_i$ for all $i \in I$. Then

$$\sum_{i \in I} \kappa_i = |\bigcup_{i \in I} \{i\} \times \kappa_i| \qquad \sum_{i \in I} \lambda_i = |\bigcup_{i \in I} \{i\} \times \lambda_i|$$

Note that $\{i\} \times \kappa_i \subseteq \{i\} \times \lambda_i$.

Define f by $f(i, \alpha) = (i, \alpha)$, so $f : \bigcup_{i \in I} \{i\} \times \kappa_i \hookrightarrow \bigcup_{i \in I} \{i\} \times \lambda_i$.
Thus

$$\sum_{i \in I} \kappa_i \leq \sum_{i \in I} \lambda_i$$

□

Infinite sums and cardinal product

☞ The following is easy to verify for cardinals κ and λ

$$\sum_{\alpha<\lambda} \kappa = \lambda \cdot \kappa.$$

(map the αth copy of κ to $\{\alpha\} \times \kappa$.)

Generalizing,

Theorem

Let λ be an infinite cardinal and κ_α nonzero for each α.

$$\sum_{\alpha<\lambda} \kappa_\alpha = \lambda \cdot \sup\{\kappa_\alpha \mid \alpha < \lambda\}$$

Proof

Proof.

Let λ be an infinite cardinal, and $\kappa = \sup\{\kappa_\alpha \mid \alpha < \lambda\}$.

☞ Since $\kappa_\alpha < \kappa$,

$$\sum_{\alpha<\lambda} \kappa_\alpha \leq \sum_{\alpha<\lambda} \kappa = \lambda \cdot \kappa.$$

☞ Conversely (using the hypothesis that $\kappa_\alpha > 0$ for each α)

- $\lambda \leq \sum_{\alpha<\lambda} \kappa_\alpha$: since $\lambda = \lambda \cdot 1 = \sum_{\alpha<\lambda} 1 \leq \sum_{\alpha<\lambda} \kappa_\alpha$;
- $\kappa \leq \sum_{\alpha<\lambda} \kappa_\alpha$: since $\kappa_\alpha \leq \sum_{\alpha<\lambda} \kappa_\alpha$ and $\kappa = \sup\{\kappa_\alpha \mid \alpha < \lambda\}$.

Recall that $\lambda \cdot \kappa = \max\{\lambda, \kappa\}$ (since λ is infinite by hypothesis); so

$$\sum_{\alpha<\lambda} \kappa = \lambda \cdot \kappa \leq \sum_{\alpha<\lambda} \kappa_\alpha.$$

□

Infinite products

The following generalizes the product of two cardinal numbers: $\kappa_1 \cdot \kappa_2$:

Definition

Let $\langle A_i \,|\, i \in I\rangle$ be a system of sets, and $\kappa_i = |A_i|$.
Then the product of $\langle \kappa_i \,|\, i \in I\rangle$ is

$$\prod_{i \in I} \kappa_i = |\prod_{i \in I} A_i|$$

- Letting $I = \{1, 2\}$, we get $\kappa_1 \cdot \kappa_2$.
- We need AC to show this definition "makes sense", so that the definition does not depend on the choice of the family $\langle A_i \,|\, i \in I\rangle$.
- The $\prod$ on the right-side of the definition is cartesian product (a set of functions), the $\prod$ on the left-side (being defined) is cardinal product (a cardinal number.)

Crucial Lemma for definition

Lemma

If $\langle A_i \,|\, i \in I\rangle$ and $\langle B_i \,|\, i \in I\rangle$ are systems of sets such that $|A_i| = |B_i|$ for every $i \in I$, then $\left|\prod_{i \in I} A_i\right| = \left|\prod_{i \in I} B_i\right|$.

Proof.

For each $i \in I$ choose a bijection $f_i : A_i \rightleftarrows B_i$.
Define f for each $a = \langle a_i \,|\, i \in I\rangle \in \prod_{i \in I} A_i$ by

$$f(a) = \langle f_i(a_i) \,|\, i \in I\rangle \in \prod_{i \in I} A_i.$$

Thus, $f : \prod_{i \in I} A_i \rightleftarrows \prod_{i \in I} B_i$ □

Infinite products: Examples

Some examples of infinite products.

☞ Verify

$$\prod_{\alpha<\lambda} \kappa = \kappa^{\lambda}$$

☞ Let $\langle \kappa_\alpha \mid \alpha < \kappa \rangle$ be a sequence of cardinals with $2 \leq \kappa_\alpha \leq 2^\kappa$. Then

$$2^\kappa = \prod_{\alpha<\kappa} 2 \leq \prod_{\alpha<\kappa} \kappa_\alpha \leq (2^\kappa)^\kappa = 2^\kappa.$$

König's Theorem

The following is a generalization of Cantor's diagonalization due to König in 1905 and independently by Zermelo in 1908.

Theorem

If $\kappa_i < \lambda_i$ for all $i \in I$, then

$$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$$

We obtain Cantor's theorem as a consequence:

$$\kappa = \sum_{\alpha<\kappa} 1 < \prod_{\alpha<\kappa} 2 = 2^\kappa$$

Proof: Part I

Fix $A_i \subseteq B_i$ (for $i \in I$) with $|A_i| = \kappa_i$ and $|B_i| = \lambda_i$, where B_is are disjoint.

$\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i$.

☞ Fix $b_i \in B_i - A_i$ for each i. Define $f : \bigcup_{i \in I} A_i \to \prod_{i \in I} B_i$ by $f(x) = \langle c_i \mid i \in I \rangle \in \prod_{i \in I} B_i$ where

$$c_i = \begin{cases} x & \text{if } x \in A_i \\ b_i & \text{o.w.} \end{cases}$$

f is well-defined since the A_is are disjoint.

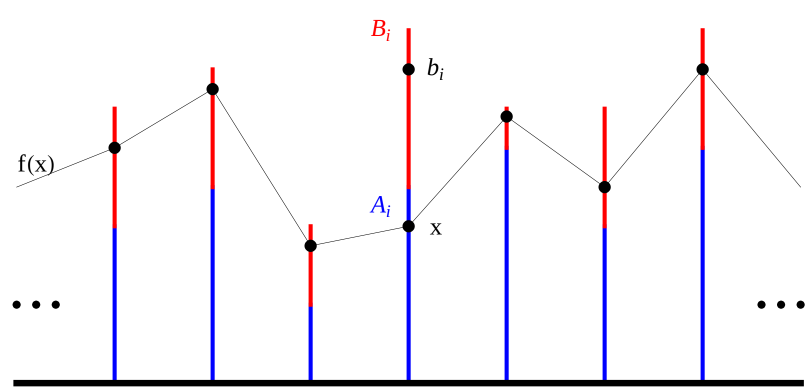
☞ f is injective: Suppose $f(x) = f(y)$. Since $x \in A_i$ for some i, $x \neq b_j$ for every $j \in I$.

So we must have $y = x$ by the definition of f.

Proof König's Theorem (I)

$$b_i \in B_i - A_i \neq \emptyset$$

$$f(x) = \begin{cases} x & \text{if } x \in A_i \\ b_i & \text{o.w.} \end{cases}$$



Proof: Part II

Fix B_i for $i \in I$ so that $|B_i| = \lambda_i$.

$\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.

☞ Let $X_i \subseteq \prod_{i \in I} B_i$ with $|X_i| = \kappa_i$ for each i. We will show that

$$\bigcup_{i \in I} X_i \neq \prod_{i \in I} B_i.$$

☞ Project X_i onto B_i:

$$A_i = \{x_i \,|\, \langle x_i \,|\, i \in I \rangle \in X_i\}.$$

Fix $b_i \in B_i - A_i$. Let $b = \langle b_i \,|\, i \in I \rangle$. Then, for each i, $b \notin X_i$ since $b_i \notin A_i$. Thus,

$$b \in \prod_{i \in I} B_i - \bigcup_{i \in I} X_i$$

Proof König's Theorem (II)

$A_i = \{x_i \,|\, \langle x_i \,|\, i \in I \rangle \in X_i\}$
$b_i \in B_i - A_i \neq \emptyset$
$b = \langle b_i \,|\, i \in I \rangle$