Math 582
Introduction to Set Theory

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Definition of Ordered Pair

Definition. An ordered pair is a collection determined by two objects a and b, which we write as (a, b), and is characterized by the following Axiom governing the identity of ordered pairs.

Axiom of Identity for Ordered Pairs. Let (a, b) and (c, d) be ordered pairs. Then the following are equivalent:

$$\begin{array}{ll} (a) & (a, b) = (c, d) \\ (b) & a = c \ \wedge \ b = d. \end{array}$$

Ordered Pair vs. Unordered Pair

☞ Let a and b be any two distinct objects (that is, $a \neq b$). Then,

$$(a, b) \neq (b, a)$$

by the Axiom of Identity for Ordered Pairs.

☞ So, ordered pairs are distinct from unordered pairs:

$$(a, b) \neq \{a, b\},$$

n-tuples

We can generalize ordered pair to finite collections.

Definition. An n-tuple (for $n \geq 2$) is a collection determined by n objects $a_0, \ldots, a_{n-1}$ defined recursively for $n > 2$ by

$$(a_0, \ldots, a_{n-1}) = ((a_0, \ldots, a_{n-2}), a_{n-1})$$

Note. An n-tuple is an ordered pair, whose first object is an $(n-1)$-tuple (when $n > 2$).

Characterizing *n*-tuples

The key property characterizing n-tuples is the following generalization from ordered pairs.

Principle. Let $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ be ordered n-tuples. Then the following are equivalent:

$$\begin{array}{ll} (a) & (a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{n-1}) \\ (b) & a_i = b_i \quad \text{for each } i < n. \end{array}$$

Characterizing *n*-tuples

Proof.

The proof is by induction on $n \geq 2$. When $n = 2$ this is just the statement of the Axiom of Identity for ordered pairs.
☞ Suppose for n (the induction hypothesis)

$$\begin{array}{ll} (a) & (a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{n-1}) \\ (b) & a_i = b_i \quad \text{for each } i < n. \end{array}$$

and we will show this equivalence extends to $(n+1)$-tuples.
Let $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ be n-tuples. Then

$$\begin{aligned} (a_0, \ldots, a_n) = (b_0, \ldots, b_n) &\leftrightarrow ((a_0, \ldots, a_{n-1}), a_n) = ((b_0, \ldots, b_{n-1}), b_n) \\ &\leftrightarrow (a_0, \ldots, a_{n-1}) = (b_0, \ldots, b_{n-1}) \ \wedge \ a_n = b_n \\ &\leftrightarrow a_i = b_i \quad \text{for each } i < n+1. \end{aligned}$$

The last line is by the induction hypothesis. □

Definition of Cartesian Product

Definition. For any two sets A and B, the cartesian product of A and B is the set denoted by

$$A \times B = \{(a, b) \mid a \in A \ \wedge \ b \in B\}.$$

In the same way, for each $n \geq 2$,

$$\begin{aligned} A_0 \times \ldots \times A_{n-1} &= \{(a_0, \ldots, a_{n-1}) \mid a_i \in A_i \text{ for each } i < n\} \\ A^n &= A \times A \times \ldots \times A \ \ (n\text{-fold}) \ . \end{aligned}$$

Function

Definition. Let A and B be nonempty sets. We use the notation

$$f : A \to B$$

to indicate that f is a function which to associates each $a \in A$ a unique $f(a) \in B$.

Assumption. For now, we will take functions to be primitive mathematical objects, just like numbers $(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C})$ and sets.

Naming Convention

☞ We will use the abbreviation $\big(x \mapsto f(x)\big)$ to talk about a function without officially naming it. For example, the function

$$\big(x \mapsto x^2\big)$$

is the function on $\mathbb{R}$ which assigns each real its square. If we name this function f, then it is defined by the formula

$$f(x) = x^2 \quad (x \in \mathbb{R})$$

Example. The function $(x \mapsto \{x\})$, mapping an object to its singleton.

Axiom for functions

☞ We take the following as an axiom of function identity.

Axiom of Function Identity. Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be functions. Then the following are equivalent.

1. $f = g$.
2. $A = C$ and $B = D$ and

$$\forall a \in A\,\big(f(a) = g(a)\big).$$

Domain and Range

Definition. Let $f : A \to B$ be a function. Then we define

- The domain of f, written $\operatorname{dom} f$, is A.
- The codomain of f is B.
- The range of f is the set

$$\operatorname{ran} f = \{f(a) \mid a \in A\}.$$

For any $X \subseteq A$, the image of X under f is the set

$$f[X] = \{f(x) \mid x \in X\};$$

and for any $Y \subseteq B$, the pre-image of Y by f is the set

$$f^{-1}[Y] = \{x \in A \mid f(x) \in Y\}.$$

Definition: Bijection

Definition. Let $f : A \to B$ be a function. Then

- f is injective if
$$\forall x, y \left(f(x) = f(y) \;\to\; x = y\right).$$
We write $f : A \hookrightarrow B$ if f is injective.
- f is surjective if
$$\forall b \in B \, \exists a \in A \left[f(a) = b\right].$$
We write $f : A \twoheadrightarrow B$ if f is surjective.
- f is bijective if f is injective and surjective.
We write $f : A \rightleftarrows B$ if f is bijective.

Definition: Composition

Definition. Let $f : A \to B$ and $g : B \to C$ be functions. Then the composition of f and g is the function

$$g \circ f : A \to C$$

defined by

$$(g \circ f)(a) = g(f(a))$$

Composition is associative

Proposition. Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be functions. Then

$$h \circ (g \circ f) = (h \circ g) \circ f \qquad \text{(Associativity)}$$

Proof. Let $a \in A$. Then

$$\begin{aligned} (h \circ (g \circ f))(a) &= h((g \circ f)(a)) \\ &= h(g(f(a))) \\ &= (h \circ g)(f(a)) \\ &= ((h \circ g) \circ f)(a) \end{aligned}$$

It follows by the Axiom of function identity that $h \circ (g \circ f) = (h \circ g) \circ f$.

Composition preserves nice properties

Proposition. Let $f : A \to B$ and $g : B \to C$ be functions. Then

(a) If f and g are injective, then $g \circ f$ is injective.

(b) If f and g are surjective, then $g \circ f$ is surjective.

(c) If f and g are bijective, then $g \circ f$ is bijective.

Definition: Inverse

Proposition. Let $f : A \leftrightarrows B$ be a bijection. Then, for each $b \in B$ there is a unique $a \in A$ such that $b = f(a)$.

Definition. Let $f : A \leftrightarrows B$ be a bijection. Then, we define the inverse function

$$f^{-1} : B \to A$$

by the condition

$$f^{-1}(b) = a \leftrightarrow b = f(a) \qquad \text{for all } a \in A \text{ and } b \in B.$$

(f^{-1} is a function by the previous Proposition.)

Note. The inverse image $f^{-1}[B]$ is the precisely the image of B under f^{-1}.

Functions and set operators

Proposition. Let $f : A \to B$ and $X, Y \subseteq A$. Then

$$(a) \qquad f[X \cup Y] = f[X] \cup f[Y].$$

If f is an injection, then

$$(b) \qquad f[X \cap Y] = f[X] \cap f[Y]$$
$$(c) \qquad f[X - Y] = f[X] - f[Y].$$

Pre-image and set operators

Proposition. Let $f : A \to B$ and $X, Y \subseteq B$. Then

$$(a) \qquad f^{-1}[X \cup Y] = f^{-1}[X] \cup f^{-1}[Y]$$
$$(b) \qquad f^{-1}[X \cap Y] = f^{-1}[X] \cap f^{-1}[Y]$$
$$(c) \qquad f^{-1}[X - Y] = f^{-1}[X] - f^{-1}[Y].$$

Infinite sequences

We extend the definition of *n*-tuple to infinite sequences

$$(a_0, a_1, \ldots) \text{ or } (a_n)_{n\in\mathbb{N}}.$$

Definition. An infinite sequence is a function whose domain is $\mathbb{N}$.

Examples.

- Let $s : \mathbb{N} \to \mathbb{R}$ be defined be $s(0) = 1$ and $s(n) = \frac{1}{n+1}$ for $n > 0$. This defines a sequence

$$s = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$$

- Let $s : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ by $s(n) = \{n\}$. This defines a sequence

$$s = (\{0\}, \{1\}, \{2\}, \ldots)$$

Principle of Infinite Sequences

Principle. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two infinite sequences. The following are equivalent

$$\begin{array}{ll} (a) & (a_n)_{n\in\mathbb{N}} = (b_n)_{n\in\mathbb{N}}. \\ (b) & a_n = b_n \quad \text{for all } n \in \mathbb{N} \end{array}$$

Proof. An easy consequence of the Axiom of Identity for Functions.

Sequences and n-tuples

Remark. We could have defined an n-tuple as a function whose domain is the set $\{0, 1, \ldots, n-1\}$. We might then write

$$(a_0, \ldots, a_{n-1}) = f \quad \text{for the function } f \text{ with } f(i) = a_i.$$

For example, if A is a set, then the cartesian product is

$$A^n = \{f \mid f : \{0, \ldots, n-1\} \to A\}$$

Then, the Identity Principle for n-tuples follows from the Axiom of Function Identity:
for any $f, g \in A^n$:

$$f = g \;\leftrightarrow\; \forall i \left[f(i) = g(i) \right].$$

Infinite Unions and Intersections

Notation. Let $\langle A_n \mid n \in \mathbb{N} \rangle$ be an infinite sequence of sets. We make the following definitions

$$\bigcup_{n=0}^{\infty} A_n = \bigcup \{X \mid \exists n\, X = A_n\}$$

$$\bigcap_{n=0}^{\infty} A_n = \bigcap \{X \mid \exists n\, X = A_n\}$$

Equivalent formulation

Our definition of infinite unions agrees with the usual understanding.

Lemma. Let $\langle A_n \mid n \in \mathbb{N}\rangle$ be an infinite sequence of sets. Then

$$\bigcup_{n=0}^{\infty} A_n = \{x \mid \exists n\, x \in A_n\}$$

$$\bigcap_{n=0}^{\infty} A_n = \{x \mid \forall n\, x \in A_n\}$$