Math 582
Intro to Set Theory
Lecture 29

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

April 1, 2009

Introduction

☞ There are two results here. First, an application of Tukey's Lemma to a proof of the Boolean Ultrafilter Theorem: every filter can be extended to an ultrafilter.

☞ The second result here is a lovely equivalence to AC due to Tarski in 1924.

☞ This lecture is intended to supplemental Hrbacek + Jech, section 8.2.

Tukey's Lemma

☞ Let $\mathcal{F} \subseteq \mathcal{P}(A)$. Then

- $\mathcal{F}$ has finite character iff for all $X \subseteq A$:
  $X \in \mathcal{F}$ iff every finite subset $X \in \mathcal{F}$.
- Tukey's Lemma is the assertion that whenever $\mathcal{F} \subseteq \mathcal{P}(A)$ has finite character and $X \in \mathcal{F}$, there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.

☞ We showed in Lecture 28 that Tukey's Lemma is equivalent to the Axiom of Choice.

Tukey's Lemma

☞ Tukey's Lemma no longer true if we replace *finite* with *countable*:

- $\mathcal{F}$ has countable character iff for all $X \subseteq A$:
  $X \in \mathcal{F}$ iff every countable subset $X \in \mathcal{F}$.
- Countable Tukey's Lemma is the assertion that whenever $\mathcal{F} \subseteq \mathcal{P}(A)$ has countable character and $X \in \mathcal{F}$, there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.

☞ Recall, $\mathcal{P}_{\text{fin}}(\omega)$ is the set of all finite subsets of ω, and has NO maximal sets. Although $\mathcal{P}_{\text{fin}}(\omega)$ does NOT have finite character, it does have countable character.

Filters and Ultrafilters

Definition

Let A be a nonempty set and $\mathcal{F} \subseteq \mathcal{P}(A)$. We say that $\mathcal{F}$ is a filter if it is nonempty, $\emptyset \notin \mathcal{F}$, and the following two properties are satisfied:

(a) $X, Y \in \mathcal{F} \ \rightarrow \ X \cap Y \in \mathcal{F}$ (closed under intersection),

(b) $X \in \mathcal{F} \ \wedge \ X \subseteq Y \ \rightarrow \ Y \in \mathcal{F}$ (upward closure)

$\mathcal{F}$ is an ultrafilter on A if it is a filter and additionally

(c) If $X \in \mathcal{P}(A)$ then either $X \in \mathcal{F}$ or $A - X \in \mathcal{F}$ (closure under complementation)

☞ $\mathcal{F}$ is said to have the finite intersection property if for each finite $\mathcal{F}' \subseteq \mathcal{F}$ we have $\bigcap \mathcal{F}' \neq \emptyset$.

Filters and Ultrafilters

Example. Let $a \in A$ and let

$$\mathcal{F}_a = \{X \subseteq A \mid a \in X\}.$$

Then $\mathcal{F}_a$ is an ultrafiler. A set of the form $\{a\}$ is called an atom of $\mathcal{P}(A)$.

Example. Let $X \subseteq A$ and let

$$\mathcal{F}_X = \{Y \subseteq A \mid X \subseteq Y\}.$$

Then $\mathcal{F}_X$ is a filter; however, it is only an ultrafilter if X is an atom.
Filters of this type are called principal filters.

Main Theorem

☞ It is known that the following theorem cannot be proven in ZF (without AC).

It is also known that this theorem is not equivalent to the AC: you cannot prove AC from the theorem together with the other axioms of ZF.

Theorem

Let A be a nonempty set and assume $\mathcal{F} \subseteq \mathcal{P}(A)$ has the finite intersection property. Then $\mathcal{F}$ can be extended to an ultrafilter:

there is an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(A)$ with $\mathcal{F} \subseteq \mathcal{U}$.

Proof

☞ Consider the set of a families $\mathcal{X} \subseteq \mathcal{P}(A)$ which have the finite intersection property. This family has finite character and $\mathcal{F}$ is a member of the family.

Tukey's Lemma implies that there is a family of subsets $\mathcal{U} \subseteq \mathcal{P}(A)$ with $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U}$ maximal among all families with the finite intersection.

☞ The rest of the proof is to show that $\mathcal{U}$ is an ultrafilter.

It is typical of applications of AC using maximality principles that the set-up for applying AC is minimal, but verifying the maximal set satisfies whatever conditions are needed is NO LONGER a problem for set theory, but purely technical.

Proof – continued

☞ $\mathcal{U}$ is a filter. Clearly, $\mathcal{U} \neq \emptyset$ and $\emptyset \notin \mathcal{U}$.

(a). Let $X, Y \in \mathcal{U}$. For any finite $\mathcal{U}' \subseteq \mathcal{U}$,

$$\emptyset \neq \bigcap \mathcal{U}' \cap (X \cap Y),$$

Since $\mathcal{U}$ hss the finite intersection property. By the maximality of $\mathcal{U}$, $X \cap Y \in \mathcal{U}$.

(b). Suppose $X \in \mathcal{U}$ and $X \subseteq Y$. Then for any finite $\mathcal{U}' \subseteq \mathcal{U}$,

$$\emptyset \neq \bigcap \mathcal{U}' \cap X \subseteq \bigcap \mathcal{U}' \cap Y.$$

Again, by the maximality of $\mathcal{U}$, we have $Y \in \mathcal{U}$.

Proof – continued

(c). Show that $\mathcal{U}$ is an ultrafilter.

☞ Let $X \subseteq A$ and $X \notin \mathcal{U}$. Then for some $\mathcal{U}_1 \subseteq \mathcal{U}$,

$$\bigcap \mathcal{U}_1 \cap X = \emptyset.$$

So,

$$\bigcap \mathcal{U}_1 \subseteq A - X.$$

Let $\mathcal{U}_2 \subseteq \mathcal{U}$ be any finite family of subsets. Since $\mathcal{U}$ is a filter

$$\emptyset \neq \bigcap \mathcal{U}_1 \cap \bigcap \mathcal{U}_2 \subseteq A - X.$$

Then $A - X \in \mathcal{U}$ by the maximality of $\mathcal{U}$.

Cartesian Products

☞ If A is infinite and well-orderable then $A \times A \approx A$ (without AC.)

☞ With AC, $A \times A \approx A$ for all infinite A.

Theorem (Tarski, 1924)

The Axiom of Choice is equivalent to the statement:

$A \times A \approx A$ *for all infinite sets* A.

Technical Lemma

☞ The following lemma is of technical interest for the proof of Tarski's Theorem.

Lemma

Let A be any set, and B be a well-ordered set with $A \cap B = \emptyset$. If $A \times B \preccurlyeq A \cup B$ then A and B are comparable (i.e. $A \preccurlyeq B$ or $B \preccurlyeq A$.)

Proof Lemma

Proof.

Let $h : A \times B \hookrightarrow A \cup B$. Two cases.

(a). For some $x \in A$, $\{x\} \times B \subseteq A$. Then, $B \preccurlyeq A$.

(b). For every $x \in A$ there is a $z \in B$ such that $h(x, z) \in B$. Define an injection $f : A \hookrightarrow B$ by

$$f(x) = h(x, z) \qquad \text{where } z \in B \text{ is least with } h(x, z) \in B.$$

Since h is injective, so is f. Thus, $A \preccurlyeq B$.

✓ Therefore, $A \preccurlyeq B$ or $B \preccurlyeq A$.

□

Proof Theorem

☞ AC implies every set is well-ordered; and, each well-ordered and infinite set satisfies $A \approx A \times A$.

☞ Suppose $A \approx A \times A$ for every infinite set A. Fix an arbitrary infinite set B. Recall that $\aleph(B)$ (Hartogs' aleph) is a von Neumann cardinal (so, a well-ordered set) satisfying $\aleph(B) \not\preccurlyeq B$. We may suppose $\aleph(B)$ and B are disjoint.

By our hypothesis (using $A = B \cup \aleph(B)$)

$$\begin{aligned} B \times \aleph(B) &\preccurlyeq (B \cup \aleph(B)) \times (B \cup \aleph(B)) && (\subseteq) \\ &\approx B \cup \aleph(B) && \text{hypothesis} \end{aligned}$$

By the previous Lemma: $B \preccurlyeq \aleph(B)$ or $\aleph(B) \preccurlyeq B$. The second is impossible; so, $B \preccurlyeq \aleph(B)$. Thus, we can well-order B using this injection.

AC in Set Theory

☞ We need the Axiom of Choice for a decent development of cardinality:

- Cardinal comparability: $A \preccurlyeq B$ or $B \preccurlyeq A$ for all sets A, B.
- Extending von Neumann cardinal to all sets. (It is possible to define $|A|$ in an intelligible way without AC, due to Dana Scott; we will look into this in the last week of class.)
- Infinite=Dedekind Infinite: Every infinite set has a countable subset.
- Cardinal exponentiation: $\kappa^\lambda = \left|{}^\lambda\kappa\right|$ requires AC to guarantee right-side even a von Neumann cardinal
- Infinite sums and products: Without the axiom of choice the size of $\left|\bigcup_{i\in I} A_i\right|$ (the A_i's disjoint) and $\left|\prod_{i\in I} A_i\right|$ depend on the sets themselves (the A_i's) and not their sizes.
- Without AC, it is possible that the the real numbers are a countable union of countable sets!! and the cartesian product of nonempty sets is nevertheless empty!!

AC in Set Theory

AC has numerous important mathematical applications, many of which have turned-out to be equivalent to AC (in **ZF**):

- Well-Ordering Principle (due to Zermelo in 1904.)
- Cardinal Comparability (due to König in 1905.)
- The Set-Theoretic Distributive Law (see Hrbacek+Jech, Exercise 8.1.11)

$$\bigcap_{i\in I}\bigcup_{j\in J} A_{i,j} = \bigcup_{f\in {}^IJ} A_{i,f(j)}$$

- $A \times A \approx A$ for every infinite set A (due to Tarski in 1924, as are the following.)
- $A \cup B \approx A \times B$ for every pair of disjoint infinite sets A and B.
- $A \prec B$ and $C \prec D$ implies $A \cup C \prec B \cup D$ for all pairwise disjoint sets A, B, C, D.
- $A \cup C \prec B \cup C$ implies $A \prec B$ for all pairwise disjoint sets A, B, C.

AC in Mathematics

AC has numerous important mathematical applications, many of which have turned-out to be equivalent to AC (in **ZF**):

- Every vector space has a basis. (Hamel, 1905; proven equivalent by Andreas Blass in 1984.)
- Tychonov's Theorem: The product of compact topological spaces is compact. (1930.)
- Every commutative ring with identity has a maximal ideal.
- Every distributive lattice has a maximal ideal.
- (Downward) Lowenheim-Skolem-Tarski Theorem: A first-order sentence with a model of cardinality κ has model of any infinite cardinality $\mu \leq \kappa$.

AC in Mathematics

The following mathematical applications are consequence of AC, but also follow from weaker choice principles: (equivalences to BPI noted)

- Boolean Prime Ideal Theorem (Homework 6, Problem 4.)
- Hahn-Banach Theorem (BPI, see Hrbacek and Jech, Example 8.2.9)
- Nielsen-Schreier Theorem: Every subgroup of a free group is free. (BPI)
- Every field has an algebraic closure. (BPI, Due to Steinitz in 1910.)
- There is a Lebesgue nonmeasurable set. (Due to Vitali in 1905, see Hrbacek and Jech, Example 8.2.13, derivable from BPI.)
- The additive group of $\mathbb{R}$ and $\mathbb{C}$ are isomorphic.
- Compactness and Completeness Theorems for First-Order Logic. (BPI)
- The Stone Representation Theorem for Boolean algebras (BPI, Marshall Stone, 1936 while at UofC.)

Note. Homework 6, Problems 8-10, introduce the the Axiom of Dependent Choices, which is a very useful axiom in the spirit of AC, but considerably weaker in strength.