Math 582
Intro to Set Theory
Lecture 28

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March 31, 2009

Introduction

- *The Axiom of Choice is necessary to select an infinite number of socks, but not necessary to select infinite number of shoes.* – Bertrand Russell
- *The axiom gets its name not because mathematicians prefer it to the other axioms.* – A.K. Dewdney
- *The Axiom of Choice is obviously true, the well-ordering principle is obviously false, and who can tell about Zorn's Lemma.* – Jerry Bona

Introduction

The lectures will be divided into three parts:

1. Statements of the Axiom of Choice, and its use in proving loose ends from our development of cardinal number. This will include some weaker principles like Axiom of Dependent Choice and Axiom of Countable Choice.
2. Statements of Maximality Principles (Zorn's Lemma, Tukey's Lemma) and their equivalence to the Axiom of Choice.

☞ The material is drawn from Hrbacek and Jech, Chapter 8.

Axiom of Choice

SING(x) says that "x is a *singleton* set":

$$\text{SING}(x) \iff \exists y \in x \forall z \in x (z = y)$$

Axiom 9. Choice

$$\emptyset \notin \mathcal{F} \wedge \forall x, y \in \mathcal{F}(x \neq y \rightarrow x \cap y = \emptyset) \quad \rightarrow \quad \exists C \forall x \in \mathcal{F}(\text{SING}(C \cap x))$$

☞ In English: whenever $\mathcal{F}$ is a disjoint family of nonempty sets, there is a set C such that $C \cap x$ is a singleton set for all $x \in \mathcal{F}$. The set C is called a Choice set for $\mathcal{F}$ because it chooses an element from each set in $\mathcal{F}$.

AC and equivalents

There are many variations on the Axiom of Choice, and equivalences. I have gathered several close variations here.

Theorem

*The following are equivalent in **ZF**:*

(1) *The Axiom of Choice.*

(2) *Every set has a choice function.*

(3) *Well-ordering Principle: Every set can be well-ordered.*

(4) *Cardinal Comparability:* $\forall x, y\,(x \preccurlyeq y \ \vee\ y \preccurlyeq x)$.

(5) *Multiplicative Principle: If* $X_i \neq \emptyset$ *for each* $i \in I$ *then* $\prod_{i \in I} X_i \neq \emptyset$.

(6) *Any surjective function has a right inverse. (That is, if* $f : A \twoheadrightarrow B$ *then there is a function* $g : B \to A$ *such that* $f \circ g(x) = x$ *for every* $x \in B$.*)*

Choice functions

☞ In practice our formulation of the Axiom of Choice is not very useful, since frequently one needs to choose elements from sets which are not disjoint. In these cases, the choosing is done by a choice functions:

Definition

A choice function for a set A is a function $g : \mathcal{P}(A) - \{\emptyset\} \to A$ such that $g(x) \in x$ for every $x \in \mathcal{P}(A) - \{\emptyset\}$.

Theorem

*The following are equivalent in **ZF**:*

(1) *The Axiom of Choice.*

(2) *Every set has a choice function.*

Proof of Theorem

Proof.

(1) $\Rightarrow$ (2). Let A be any set and let $\mathcal{F} = \{\{x\} \times x \mid x \in \mathcal{P}(A) - \{\emptyset\}\}$. If $x \neq y$ then $\{x\} \times x \cap \{y\} \times y = \emptyset$.

Let C be a choice set for $\mathcal{F}$; then $C \cap \{x\} \times x = \{(x, i)\}$ where $i \in x$. So, $C : \mathcal{P}(A) - \{\emptyset\} \to A$ is a choice function.

(2) $\Rightarrow$ (1). Let $\mathcal{F}$ be a disjoint family of non-empty sets, and let $A = \bigcup \mathcal{F}$. Let g be a choice function for A, and let $C = \{g(x) \mid x \in \mathcal{F}\}$.

So, $C \cap x = \{g(x)\}$ for every $x \in \mathcal{F}$; thus, C is a choice set.

□

Well-Ordering and AC

Zermelo originally (1904) introduced the Axiom of Choice in order to prove that every set is well-orderable; this latter was needed by Cantor to prove the comparability of the sizes of sets:

Theorem

*The following are equivalent in **ZF**:*

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(2) *Every set has a choice function.*

(3) *Well-ordering Principle: Every set can be well-ordered.*

(4) *Cardinal Comparability:* $\forall x, y \left(x \preccurlyeq y \ \vee \ y \preccurlyeq x\right)$.

(5) *Multiplicative Axiom: If* $X_i \neq \emptyset$ *for each* $i \in I$ *then* $\prod_{i \in I} X_i \neq \emptyset$.

Proofs of equivalences

(2) $\Rightarrow$ (3). Let g be a choice function for A. Let $\kappa = \aleph(A)$ (so, $\kappa \not\preccurlyeq A$.) Let $\text{STOP} \notin A$. Define $f : \kappa \to A \cup \{\text{STOP}\}$ so that

$$f(\alpha) = \begin{cases} g(A - \{f(\xi) \,|\, \xi < \alpha\}) & \text{if } A - \{f(\xi) \,|\, \xi < \alpha\} \neq \emptyset \\ \text{STOP} & \text{o.w.} \end{cases}$$

Notice that $f(\xi) \neq f(\alpha)$ when $\xi < \alpha$ and $f(\alpha) \neq \text{STOP}$.

☞ Since $\kappa \not\preccurlyeq A$ there can be no injection $\kappa \hookrightarrow A$; so, there must be an $\alpha < \kappa$ with $f(\alpha) = \text{STOP}$.

✓ Thus, $f \restriction \alpha : \alpha \rightleftarrows A$, so that A has a well-ordering of order type α.

Proofs of equivalences

(3) $\Rightarrow$ (2). Suppose A is well-orderable, and fix a well-ordering $R \subseteq A \times A$. Define a choice function g on $\mathcal{P}(A) - \{\emptyset\}$ by letting $g(x)$ be the R-least element of x (where $x \subseteq A$ is nonempty.)

(3) $\Rightarrow$ (4). Fix sets x, y. Then x and y are well-orderable, so that $|x|$ and $|y|$ are von Neumann cardinals. But $|x| \leq |y|$ or $|y| \leq |x|$. (4) follows since $u \preccurlyeq v$ iff $|u| \leq |v|$ for any well-orderable sets u and v.

(4) $\Rightarrow$ (3). Let $\kappa = \aleph(A)$, so that $\kappa \not\preccurlyeq A$; but, then $A \preccurlyeq \kappa$ by (4). So, A is well-orderable. (If $f : A \hookrightarrow \kappa$ then define xRy iff $f(x) < f(y)$ for $x, y \in A$; this is a well-ordering of A.)

Proofs of equivalences

(5) $\Rightarrow$ (2). Let A be arbitrary and set the index $I = \mathcal{P}(A) - \{\emptyset\}$, so that $X_D = D$ for every $D \in I$.

Then by (5) there is some $f \in \prod_{D \in I} X_D \neq \emptyset$; but, $f(X_D) \in X_D = D$, so f is a choice function.

(2) $\Rightarrow$ (5). Let $\langle X_i \mid i \in I \rangle$ be a nonempty family of sets.

Let g be a choice function for $A = \bigcup_{i \in I} X_i$; and define $f \in \prod_{i \in I} X_i$ by

$$f(i) = g(X_i).$$

Since $g(X_i) \in X_i$, it follows that $f \in \prod_{i \in I} X_i$, so $\prod_{i \in I} X_i \neq \emptyset$.

Chronology of AC

- 1904: Zermelo explicitly formulates AC and uses it to prove the Well-Ordering Theorem. This raises a storm of controversy, especially with the French Analysts (Baire, Borel, Lebesgue).
- 1904: Bertrand Russell recognizes AC as his Multiplicative Axiom.
- 1908: Zermelo provides first explicit presentation of an axiom system for set theory, and provides a rigorous proof of the equivalence of the Well-Ordering Theorem and AC.
- 1924: Building on work of Felix Hausdorff a decade earlier, Tarski and Banach derive from AC their paradoxical decomposition of the sphere: Any solid sphere can be decomposed into finitely many peices and reassembled into two solid spheres of the same size.
- 1938: Gödel establishes the relative consistency of AC with the other axioms ZF of set theory. (AC cannot be refuted by the other axioms.)
- 1964: Paul Cohen proves the independence of AC from the other axioms ZF of set theory. (AC cannot be proven from the other axioms.)

Maximal Principles

☞ The Maximal Principles assert that certain conditions are sufficient to ensure that a partially ordered set contains at least one maximal element. They are often used in place of the Axiom of Choice in texts on analysis, algebra, and topology, since they do not require use of ordinals or transfinite induction/recursion in their application.

Theorem

*The following are equivalent in **ZF**:*

(1) *The Axiom of Choice.*

(A) *Tukey's Lemma*

(B) *Hausdorff Maximality Principle*

(C) *Zorn's Lemma:*

Maximality

Definition

Let $\mathcal{F} \subseteq \mathcal{P}(A)$. Then X is maximal in $\mathcal{F}$ iff it is maximal w.r.t. $\subsetneq$; that is X is not a proper subset of any set in $\mathcal{F}$.

- If $\mathcal{F} = \mathcal{P}(A)$ then A is maximal in $\mathcal{F}$.
- If $\mathcal{F} = \mathcal{P}_{fin}(A)$ (the finite subsets of A) and A is infinite then $\mathcal{F}$ has no maximal element.
- Let A is a vector space over some field. Define $\mathcal{F} \subseteq \mathcal{P}(A)$ by $X \in \mathcal{F}$ iff X is a linearly independent set. Then X is maximal in $\mathcal{F}$ iff X is a basis. (X is linearly independent iff there is no finite $\{x_1, \ldots, x_n\} \subseteq X$ and non-zero scalars $a_1, \ldots, a_n$ such that $a_1x_1 + \ldots + a_nx_n = 0$; X is a basis if there is no linearly independent set Y with $Y \supset X$.)

☞ The significant property about this last example is that the family $\mathcal{F}$ has finite character.

Finite character and Tukey's Lemma

Definition

Let $\mathcal{F} \subseteq \mathcal{P}(A)$. Define

- $\mathcal{F}$ has finite character iff for all $X \subseteq A$:
  $X \in \mathcal{F}$ iff every finite subset $X \in \mathcal{F}$.
- Tukey's Lemma is the assertion that whenever $\mathcal{F} \subseteq \mathcal{P}(A)$ has finite character and $X \in \mathcal{F}$, there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.

Vector Spaces. The family $\mathcal{F}$ of linearly independent subsets of a vector space A is an example of a family of sets with finite character. (To verify a $X \subseteq A$ is linearly independent requires only looking at finite subsets of A.)

Tukey's Lemma implies that every linearly independent set can be expanded to a basis.

Proof of Tukey's Lemma

Theorem

The following are equivalent:

(1) *The Axiom of Choice.*

(A) *Tukey's Lemma: For any set A, if $\mathcal{F} \subseteq \mathcal{P}(A)$ is a family of sets which has finite character, then for any $X \in \mathcal{F}$ there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.*

We will prove Tukey's Lemma (A) from the Well-ordering Theorem (3); then prove the Axiom of Choice (1) from Tukey's Lemma (A).

Proof of Tukey's Lemma from Well-ordering

(3) $\implies$ (A). Let $\mathcal{F} \subseteq \mathcal{P}(A)$ be a family of sets which has finite character. Fix a well-ordering of A as $\{x_\alpha \mid \alpha < |A|\}$. Let $X \in \mathcal{F}$. Define a sequence of sets by transfinite recursion $\langle Y_\beta \mid \beta < |A|\rangle$ as follows:

1. $Y_0 = X$,
2. $Y_{\alpha+1}$ is $Y_\alpha \cup \{x_\alpha\}$ if $Y_\alpha \cup \{x_\alpha\} \in \mathcal{F}$ and $Y_{\alpha+1} = Y_\alpha$ otherwise.
3. $Y_\gamma = \bigcup\{Y_\xi \mid \xi < \gamma\}$ if γ is a limit ordinal.

☞ $Y_\beta \in \mathcal{F}$: By transfinite induction on β. For $\beta = 0$ or β a successor this is by definition. For β a limit, we assume (i.h.) that $Y_\xi \in \mathcal{F}$ whenever $\xi < \beta$. Let $P \subseteq Y_\beta$ be a finite set, so that there is a $\xi < \beta$ for which $P \subseteq Y_\xi$. But, $\mathcal{F}$ has finite character, so that $P \in \mathcal{F}$. Thus, every finite subset Y_β is in $\mathcal{F}$, so that $Y_\beta \in \mathcal{F}$ (as $\mathcal{F}$ has finite character.)

☞ $Y = Y_{|A|}$ is maximal: for each x_α if $x_\alpha \notin Y$ then this is because $Y_\alpha \cup \{x_\alpha\} \notin \mathcal{F}$. So, $Y \cup \{x_\alpha\} \notin \mathcal{F}$ by the finite character of $\mathcal{F}$.

Proof of Tukey's Lemma from Well-ordering

(A) $\implies$ (1).

☞ Let $\mathcal{F}$ be any family of disjoint nonempty sets. We need to produce a choice set C which meets every set $X \in \mathcal{F}$ in a singleton. Let $A = \bigcup \mathcal{F}$. Let $\mathcal{G}$ be the set of all partial choice sets: $D \in \mathcal{G}$ iff $D \subseteq \mathcal{P}(A)$ and $|D \cap X| \leq 1$ for every $X \in \mathcal{F}$.

☞ $\mathcal{G}$ has finite character. $D \notin \mathcal{G}$ iff there is some two element subset $D' \subseteq D$ and element $X \in \mathcal{F}$ with $|D' \cap X| = 2$.

☞ Since $\emptyset \in \mathcal{G}$, by Tukey's Lemma there is a maximal $C \in \mathcal{G}$. C is a choice set. Suppose $C \cap X = \emptyset$ for some $X \in \mathcal{F}$. Since $X \neq \emptyset$, let $u \in X$ and consider $C' = C \cup \{u\}$. This is a proper extension of C, and we must have $C' \in \mathcal{G}$ since $C \cap X = \emptyset$ and X is the unique set for which $u \in X$. This contradicts the maximality of C.

✓ The maximal $C \in \mathcal{G}$ is a choice set.

Statement of Zorn's Lemma

Definition

Let $<$ be a strict partial order on a set A. Then $C \subseteq A$ is a chain if C is totally ordered by $<$; C is a maximal chain if C is a chain and there are no chains $X \supsetneq C$.

- The Hausdorff Maximal Principle is the assertion that when $<$ is a strict partial order on a set A, there is a $\subseteq$-maximal chain $C \subseteq A$.
- Zorn's Lemma is the assertion that whenever $<$ is a strict partial order on a set A such that every chain is bounded in A:
  - (❖) For all chains $C \subseteq A$ there is some $b \in A$ such that $b \geq x$ for all $x \in C$.

  then for all $a \in A$, there is a $<$-maximal $b \in A$ with $b \geq a$.

Proof of Maximality Equivalences

Theorem

The following are equivalent:

(A) *Tukey's Lemma: For every set A, if $\mathcal{F} \subseteq \mathcal{P}(A)$ is a family of sets which has finite character, then for any $X \in \mathcal{F}$ there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.*

(B) *The Haudorff Maximality Principle: If $(A, <)$ is a partially ordered set, then there is a $\subseteq$-maximal chain $C \subseteq A$.*

(C) *Zorn's Lemma: If $(A, <)$ is a partially ordered set, such that every chain is bounded in A:*

  (❖) *For all chains $C \subseteq A$ there is some $b \in A$ such that $b \geq x$ for all $x \in C$. (Every chain is bounded in A.)*

  then for all $a \in A$, there is a $<$-maximal $b \in A$ with $b \geq a$.

Proof of Equivalences

$(A) \Rightarrow (B)$. Let $\mathcal{F} = \{C \subseteq A \mid C \text{ a chain }\}$. Notice that $C \in \mathcal{F}$ iff for $\forall\{x,y\} \subseteq C$, $x \leq y$ or $y \leq x$; so that $\mathcal{F}$ has finite character. But Tukey's Lemma $\mathcal{F}$ has a maximal member, a maximal chain as required by (B).

$(B) \Rightarrow (C)$. Let $(A,<)$ be a partially ordered set satisfying

(❖) For all chains $C \subseteq A$ there is some $b \in A$ such that $b \geq x$ for all $x \in C$. (Every chain is bounded in A.)

Fix $a \in A$ and let $A_a = \{b \in A \mid a \leq b\}$. Verify that $(A_a,<)$ is a partial order satisfying (❖) as well. By (B) there is a maximal chain $C \subseteq A_a$. By (❖) there is an element $b \in A_a$ with $b \geq x$ for all $x \in C$. We show b is maximal.

☞ Suppose b is not maximal: let $d > b$, so $d \geq x$ for all $x \in C$; but then $C \cup \{d\}$ is a chain properly extending C, contradicting the maximality of C. Thus, b is maximal in A.

Proof of Equivalences

$(C) \Rightarrow (A)$. Let $\mathcal{F}$ be a family of sets with finite character (so that $X \in \mathcal{F}$ iff every finite $P \subseteq X$ is in $\mathcal{F}$. Also, $\mathcal{F}$ is partially ordered. Consider any chain $\mathcal{C} \subseteq \mathcal{F}$ (so totally ordered by $\subset$.)

☞ $\bigcup\mathcal{C} \in \mathcal{F}$: If $P \in \bigcup\mathcal{C}$ is any finite subset, then for some $X \in \mathcal{C}$, $P \subseteq X$ so that $P \in \mathcal{F}$ by finite character. Thus, $\bigcup\mathcal{C} \in \mathcal{F}$ follows from the finite character of $\mathcal{F}$.

☞ Since $\bigcup\mathcal{C}$ is an upperbound of $\mathcal{C}$ ($X \subseteq \bigcup\mathcal{C}$ for any $X \in \mathcal{C}$), (❖) is true for $\mathcal{F}$.

☞ Let $X \in \mathcal{F}$. By Zorn's Lemma, $\mathcal{F}$ has a maximal element Y with $Y \supseteq X$. This is what we had to show to complete (A).

Chronology of Maximal Principles

- 1909: Hausdorff first explicitly formulates a maximal principle and derives it from AC.
- 1914: Hausdorff's *Grundgezüge der Mengenlehre* (the first text on set theory and general topology) includes several maximal principles.
- 1922: Kuratowski formulates and employs several maximal principles specifically to avoid transfinite ordinals.
- 1935: Max Zorn publishes his definitive version, Zorn's Lemma, as an axiom (not a lemma!!) with the hope that it would supersede cumbersome applications in algebra of transfinite induction (which had become known as transcendental methods.) Unbeknownst to Zorn, Artin formulated the lemma and proved its equivalence to AC in 1933.
- 1939-40: Teichmüler, Bourbaki and Tukey independently formulated Zorn's Lemma in terms of properties of a "finite character".