Math 582
Intro to Set Theory
Lecture 21

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March 26, 2009

Introduction

☞ We next turn to cardinal arithmetic, defining cardinal addition and multiplication, and investigate their basic properties. Our main result here is that cardinal addition and multiplication can be trivially characterized.

☞ This material comes from Hrbacek and Jech, Sections 5.1 and Chapter 7.

Cardinal addition and multiplication defined

Definition

Let α, β ordinals. Then

- $\aleph_\alpha + \aleph_\beta = |\omega_\alpha + \omega_\beta|$.
- $\aleph_\alpha \cdot \aleph_\beta = |\omega_\alpha \cdot \omega_\beta|$.

Note. A cardinal operation (on the left) is being defined by an ordinal operation (on the right.)

Convention.

- We typically use κ, λ as ranging over cardinals. There is a potential confusion about whether $\kappa + \lambda$ means the cardinal or ordinal operation. Context will determine which is meant.
- Alephs are used when the subscript matters, or when a specific cardinal is specified, such as $\aleph_0$ or $\aleph_1$.

Order lemmas

Lemma

Let $\kappa, \lambda, \kappa', \lambda'$ be cardinals, and $\kappa \leq \kappa'$ and $\lambda \leq \lambda'$. Then

$$\begin{aligned} \kappa + \lambda &\leq \kappa' + \lambda' \\ \kappa \cdot \lambda &\leq \kappa' \cdot \lambda' \end{aligned}$$

Proof.

Note that $\kappa \subseteq \kappa'$ and $\lambda \subseteq \lambda'$, so

$$\begin{aligned} \{0\} \times \kappa \cup \{1\} \times \lambda &\subseteq \{0\} \times \kappa' \cup \{1\} \times \lambda' \\ \kappa \times \lambda &\subseteq \kappa' \times \lambda' \end{aligned}$$

□

Ordinal vs. Cardinal operations

The following is immediate from the definitions

Lemma

For any ordinals α, β*,*

$$\begin{aligned} |\alpha| + |\beta| &= |\alpha + \beta| \\ |\alpha| \cdot |\beta| &= |\alpha \cdot \beta| \end{aligned}$$

Note. ω, $\omega + \omega$ and $\omega \cdot \omega$ are three different ordinals. But

$$|\omega| = |\omega + \omega| = |\omega| + |\omega| = |\omega \cdot \omega| = |\omega| \cdot |\omega|$$

Basic properties of Cardinal operations

Cardinal addition and multiplication satisfy many of the usual laws as the ordinary arithmetic operations: let κ, λ, μ be cardinals,

$$\begin{aligned} \kappa + \lambda &= \lambda + \kappa & \kappa \cdot \lambda &= \lambda \cdot \kappa \\ \kappa + (\lambda + \mu) &= (\kappa + \lambda) + \mu & \kappa \cdot (\lambda \cdot \mu) &= (\kappa \cdot \lambda) \cdot \mu \\ \kappa \cdot (\lambda + \mu) &= \kappa \cdot \lambda + \kappa \cdot \mu \end{aligned}$$

The last is by $A \times (B \cup C) = A \times B \cup A \times C$.

Cardinal addition and multiplication trivial

Cardinal sums and products are used mainly for making general statements, such as the equalities below:

Theorem

Let κ be an infinite cardinal. Then $|\kappa \cdot \kappa| = \kappa$.

Corollary

Let κ and λ be infinite cardinals, and at least one of them infinite; then

$$\begin{aligned} \kappa + \lambda &= \max\{\kappa, \lambda\} \\ \kappa \cdot \lambda &= \max\{\kappa, \lambda\} \end{aligned}$$

Proof of Corollary. Suppose $1 < \kappa \leq \lambda$. Then

$$\lambda \leq \kappa + \lambda \leq 2 \cdot \lambda \leq \kappa \cdot \lambda \leq \lambda \cdot \lambda \leq \lambda$$

Motivation: Cartesian product of countable sets

☞ We proved in Lecture 25 (Slide 20) that $\aleph_0 \cdot \aleph_0 = \aleph_0$, but the proof there used a coding into ω that was special to ω. Let's consider an alternative strategy that does generalize.

☞ Define an alternative ordering $\lhd$ on $\omega \times \omega$ as follows: $(n, m) \lhd (n', m')$ iff $\max\{n, m\} < \max\{n', m'\}$, or $\max\{n, m\} = \max\{n', m'\}$ and (n, m) preceeds (n', m') in lexicographic order. So,

$$(1, 27) \lhd (2, 27) \lhd (27, 1) \lhd (1, 28)$$

It is straightforward to verify $W = (\omega \times \omega, \lhd)$ is well-ordered.

☞ Notice that $n \times n = \{(k, \ell) \mid \max\{k, l\} < n\}$, and that under the $\lhd$ ordering, $n \times n$ is an initial segment of $\omega \times \omega$ and is also finite. (It is **not** true that $n \times n$ is an initial segment of $\omega \times \omega$ in lexicographic order.) Thus, $\text{type}(W_n) < \omega$. But, $W = \bigcup_n W_n$, so

$$\text{type}(\omega \times \omega, \lhd) = \sup\{\text{type}(W_n) \mid n < \omega\} = \omega.$$

(The first equality is by Lecture 22, slide 22.)

Proof of theorem: order defined

☞ Let κ be an infinite cardinal. We define a well-ordering of $\kappa \times \kappa$ so that type$(\kappa \times \kappa) = \kappa$. The proof the ordering works is by transfinite induction; it will be convenient to define an ordering $\lhd$ on **ON** $\times$ **ON** and prove that type$(\kappa \times \kappa) = \kappa$ when κ is a cardinal.

☞ For ordinals $\alpha, \beta, \alpha', \beta$; define

$(\alpha, \beta) \lhd (\alpha', \beta')$ iff either (i) max$\{\alpha, \beta\} <$ max$\{\alpha', \beta'\}$ or (ii) max$\{\alpha, \beta\} <=$ max$\{\alpha', \beta'\}$, and (α, β) preceedes (α', β') lexicographically.

☞ For the rest of this proof $\lhd$ is understood as the order for type$(\kappa \times \kappa)$.

Cardinal arithmetic

Proof of theorem: well-ordering

☞ $(\alpha, \beta) \lhd (\alpha', \beta')$ iff either (i) max$\{\alpha, \beta\} <$ max$\{\alpha', \beta'\}$ or (ii) max$\{\alpha, \beta\} <=$ max$\{\alpha', \beta'\}$, and (α, β) preceedes (α', β') lexicographically.

☞ Verify $\lhd$ is a total order on **ON** $\times$ **ON** (exercise!)

☞ $\lhd$ well-orders **ON** $\times$ **ON**.

Let X be a set of ordered pairs of ordinals, and δ the least ordinal such that some $(\xi, \eta) \in X$ and max$\{\xi, \eta\} = \delta$. Let $X_{=\delta}$ be the set of $(\xi, \eta) \in X$ with max$\{\xi, \eta\} = \delta$;.

X has a $\lhd$-least element iff $X_{=\delta}$ has a $\lhd$-least element.

The *lexicographic order* does well-order $X_{=\delta}$, so that if (ξ, η) is lexicographically least in $X_{=\delta}$, it is also $\lhd$-least in $X_{=\delta}$ as well.

Proof of theorem: transfinite induction

☞ We prove that type$(\kappa \times \kappa) = \kappa$ for all infinite cardinals, by transfinite induction on κ. (More formally, the induction is on the index ξ in $\aleph_\xi$.)

☞ The basis case is $\aleph_0 = |\omega|$, which was previously shown.

☞ Suppose that for all ordinals $\alpha < \kappa$, that (i.h.) type$(|\alpha| \times |\alpha|) = |\alpha|$. An important consequence of the i.h. is that for any $\alpha < \kappa$,

$$|\alpha \times \alpha| \approx |\alpha| \times |\alpha| \approx |\alpha|.$$

(The first $\approx$ is always true; the second $\approx$ is by the i.h.)

☞ It is always true that $\kappa \leq$ type$(\kappa \times \kappa)$.

☞ Suppose $\kappa <$ type$(\kappa \times \kappa)$. Then for some $\alpha < \kappa$ and $\kappa \leq$ type$(\alpha \times \alpha)$ ($\alpha \times \alpha$ is an initial segment(!) of $\kappa \times \kappa$.) But by i.h.

$$\kappa \preccurlyeq |\alpha \times \alpha| \approx |\alpha| \times |\alpha| \approx |\alpha| \prec \kappa \ \lightning$$

✓ Therefore, $\kappa =$ type$(\kappa \times \kappa)$; so, $\kappa = |\kappa \times \kappa|$.