Math 582
Intro to Set Theory
Lecture 26

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Introduction

☞ We introduce the von Neumann cardinal numbers, which are a subclass of the ordinals. Any set which can be well-ordered will be assigned a cardinal number. The assignment provides a complete invariant for the size of sets which can be well-ordered. We also prove the existence of uncountable ordinals, Hartog's Theorem

☞ This material comes from Hrbacek and Jech, Sections 5.1 and Chapter 7.

Cantor on Cardinal

Here is Cantor on cardinal number in 1895:

> *Every set A has a definite 'power', which I will also call its 'cardinal number'.*
>
> *We will call by the name 'power' or 'cardinal number' of A the general concept, whch by means of our active faculty of thought, arises from the set A when we make abstraction of its various elements x and of the order they are given.*
>
> *We denote the result of this double act of abstraction, the cardinal number or power of A by $\overline{\overline{A}}$. Since every element x, if we abstract from its nature becomes a 'unit', the cardinal number $\overline{\overline{A}}$ is a definite set composed of units, and this number has existence in our minds as an intellectual image or projection the given set A.*

Properties of cardinal numbers

One of the most difficult, yet intuitive mathematical notion to represent faithfully in set theory is that of cardinal number. From Cantor's discussion on the previous slide (and elsewhere) we can identify several basic properties of the abstraction giving rise to cardinals:

1. $A \approx \overline{\overline{A}}$,
2. $A \approx B$ if and only if $\overline{\overline{A}} = \overline{\overline{B}}$,
3. $A \preccurlyeq B$ if and only if $\overline{\overline{A}} \preccurlyeq \overline{\overline{B}}$,
4. For any two sets A and B, $\overline{\overline{A}} \preccurlyeq \overline{\overline{B}}$ or $\overline{\overline{B}} \preccurlyeq \overline{\overline{A}}$. (Cardinal Comparability)

Cardinal defined

Definition

A von Neumann cardinal is an ordinal number κ such that $\alpha \prec \kappa$ for all $\alpha < \kappa$.

- Every natural number is a cardinal number by the Pigeonhole Principle.
- ω is a cardinal. (Theorem on Slide 11 of Lecture 25.)
- Every infinite cardinal is a limit ordinal:
  $\delta + 1 \rightleftarrows \delta$ by $\delta \mapsto 0$, $n \mapsto n + 1$ and $\xi \mapsto \xi$ for $\xi \geq \omega$.
- If A is a set of cardinal numbers then $\sup A$ is a cardinal.
  If $\xi < \sup A$, then $\xi < \kappa$ for some $\kappa \leq \sup A$; so $\xi \prec \kappa \preccurlyeq \sup A$.

Well-orderable sets

Definition

A set A is well-orderable if there is a relation $R \subseteq A \times A$ such that (A, R) is a well-ordered set.

Theorem

If A is well-orderable then there is a cardinal k with $A \approx \kappa$.

Proof.

Let $R \subseteq A \times A$ so that (A, R) is well-ordered, and let $\alpha = \text{type}(A, R)$.
So, $A \approx \alpha$. Let $\kappa \leq \alpha$ be the least ordinal with $A \approx \kappa$.
κ is a cardinal number: since $\beta < \kappa$ implies $\beta \prec A \approx \kappa$. □

Cardinal number

Definition

If A is a well-orderable set the $|A|$ is the unique cardinal κ with $A \approx \kappa$. We write $|A| = \kappa$.

Lemma

Let A and B be well-orderable. Then

(a) $A \preccurlyeq B$ *iff* $|A| \leq |B|$.

(b) $A \approx B$ *iff* $|A| = |B|$.

(c) $A \prec B$ *iff* $|A| < |B|$.

(d) $A \prec B$ *or* $B \prec A$ *or* $A \approx B$.

(e) *If* $f : A \twoheadrightarrow X$ *then* X *is well-orderable and* $|X| \leq |A|$.

Proof: Part (e)

Proof of (e). Let $f : A \twoheadrightarrow X$ and (A, R) be a well-ordered set.

☞ Define $S \subseteq X \times X$ by

xSy iff aRb where a is R-least with $f(a) = x$ and b is R-least with $f(b) = y$.

☞ It is straightforward to verify that (X, S) is a well-ordered set, and $\text{type}(X, S) \leq \text{type}(A, R)$.

Uncountable cardinal numbers

☞ We have produced uncountable sets, such as $\mathcal{P}(\omega)$, but we have not yet produced an uncountable cardinal. (Recall from Lecture 25 that ordinal addition, multiplication, and exponentiation does not lead to uncountable ordinals.)

Theorem (Hartogs, 1915)

For every set A there is a cardinal κ with $\kappa \not\preccurlyeq A$.

Note. It is possible to prove this result in Z^-, Zermelo's set theory without AC, Replacement or Foundation; Hartogs proved the theorem before Replacement and Foundation were added, and before von Neuman defined the ordinals. The proof here uses Replacement for convenience.

Proof of Hartogs theorem

Proof.

☞ Let W be the set of pairs (X, R) with $X \in \mathcal{P}(A)$ and $R \subseteq \mathcal{P}(X \times X)$ where R well-orders X.

W is the set of all well-orderable subsets of A.

☞ Observe, that $\alpha \preccurlyeq A$ iff $\alpha = \text{type}(X, R)$ for some $(X, R) \in W$ (by part (e) on Slide 9.) Let

$$\beta = \{\text{type}(X, R) \,|\, (X, R) \in W\}$$

by Replacement and Comprehension.

☞ β is an ordinal; and if $\alpha \preccurlyeq A$ then $\alpha < \beta$. So, $\beta \not\preccurlyeq A$.

✓ Let $\kappa = |\beta|$; so, $\kappa \not\preccurlyeq A$. (Actually, β is already a cardinal! ☺)

□

Hartog's aleph

Definition

Define $\aleph(A)$ to be the least cardinal κ with $\kappa \not\preccurlyeq A$. (Called Hartogs aleph function.) For ordinals α it is standard to write α^+ for $\aleph(\alpha)$. (Hrbacek and Jech write $h(A)$ for the Hartogs number of A, see Definition 7.1.5.)

- For ordinals α, α^+ is the least cardinal greater than α.
- $\aleph(A)$ is most frequently used when working without AC.
- Under AC, every set is well-orderable, so $|A|$ is defined and $\aleph(A) = |A|^+$, which is the standard notation.

Initial ordinals and alephs

Definition

The initial ordinals ω_ξ are defined by recursion on ξ:

$$\begin{aligned} \omega_0 &= \omega \\ \omega_{\xi+1} &= \omega_\xi^+ \\ \omega_\eta &= \sup\{\omega_\xi \mid \xi < \eta\} \qquad \text{for limit ordinal } \eta \end{aligned}$$

We define the alephs by letting $\aleph_\xi = \omega_\xi$.

Note. We use "$\aleph_\xi$" when talking about cardinalities and "ω_ξ" when talking about order types. This distinction will be especially important when we define arithmetic operations on cardinal number, which is not the same as the ordinal arithmetic operations.

All cardinals are alephs

Lemma

(a) $\xi < \eta$ *implies* $\aleph_\xi < \aleph_\eta$.

(b) κ *is an infinite cardinal iff* $\kappa = \aleph_\xi$ *for some* ξ.

☞ A consequence of (a) and the definition of $\aleph$ is that $\aleph$ is a normal function. So, there are fixed points:

Corollary

For any ordinal α *there is an ordinal* $\beta > \alpha$ *such that* $\beta = \aleph_\beta$.

Proof: All cardinals are alephs

(a). Fix ξ. Show by transfinite induction on $\eta > \xi$ that $\aleph_\xi < \aleph_\eta$.

☞ $\eta = \delta + 1$. Assume (i.h.) $\aleph_\xi \leq \aleph_\delta$. Then

$$\aleph_\xi \leq \aleph_\delta < (\aleph_\delta)^+ = \aleph_{\delta+1} = \aleph_\eta.$$

☞ η a limit. Assume (i.h.) $\aleph_\xi < \aleph_\gamma$ whenever $\xi < \gamma < \eta$.
Since $\xi < \eta$ there is some $\xi < \gamma < \eta$, so

$$\aleph_\xi < \aleph_\gamma \leq \sup\{\aleph_\delta \,|\, \delta < \eta\} = \aleph_\eta.$$

Note. It now follows that $\aleph : \mathbf{ON} \to \mathbf{ON}$ is a normal function. So, $\alpha \leq \aleph_\alpha$ for all ordinals of α.

Proof: All cardinals are alephs

(b). Let κ be an infinite cardinal.

Since $\aleph$ is normal and $\kappa \leq \aleph_\kappa < \alpha_{\kappa+1}$, there is a unique ordinal δ such that $\aleph_\delta \leq \kappa < \aleph_{\delta+1}$.

☞ $\aleph_\delta = \kappa$: since $\aleph_\delta \leq \kappa < \aleph_{\delta+1} = (\aleph_\gamma)^+$, and there are no cardinals between $\aleph_\delta$ and $(\aleph_\delta)^+$.