Math 582
Intro to Set Theory
Lecture 25

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Introduction

☞ The main focus of this lecture will define and discuss properties of finite and countable sets. This material is covered in Hrbacek and Jech, Sections 4.2 (finite) and 4.3 (countable).

The key results are

- ♠ The Pigeonhole principle (there is no embedding of a natural number into a smaller natural number.)
- ♠ The finite sets are closed under all our set theoretic constructions: pairing, unions, cartesian products, powerset, ranges of functions.
- ♠ We will be able to prove that the countable sets are closed under finite unions, finite cartesian products, and ranges of functions.
- ♠ It looks like countable sets should be closed under countable unions and countable cartesian products, but we cannot prove this with the axioms we have.

Definition

Definition

A set is finite if it is equipotent (has same size or cardinality) to some natural number $n \in \omega$. A set is infinite if it is not finite.

☞ Every finite ordinal is a finite set.
This was the point of Exercise 5 from Homework 8.

Pigeonhole Principle

☞ The intuitive picture of the next theorem is that if you try to fit n 🐦 into m roosts and $n > m$ then some 🐦 will have to roost together.

Theorem (Pigeonhole Principle)

There is no injection from a natural number to a proper subset.

Proof of PHP

Call an injection $f : A \hookrightarrow B$ nontrivial if $B \subsetneq A$. We will prove by induction on n that there are no nontrivial injections on n.

☞ Basis. There are no nontrivial injections on $\emptyset$.

☞ Inductive. There are no nontrivial injections on n (i.h.).
Suppose that there is a nontrivial injection on $n+1$:

$f : n+1 \hookrightarrow X$ and $X \subsetneq n+1$.

Two cases: $n \notin X$ or $n \in X$.

✦ $n \notin X$.

Then $X \subseteq n$, so that $f \upharpoonright n : n \xrightarrow{1-1} X - \{f(n)\}$ and $X - \{f(n)\} \subsetneq X \subseteq n$.
Thus, $f \upharpoonright n$ is a nontrivial injection on n. $\lightning$

Proof of PHP – continued

✦✦ $n \in X$.

Let $f(i) = n$ where $i \leq n$. The goal is to construct a function $g : n \hookrightarrow X - \{n\}$; since $X \subsetneq n+1$ and $n \in X$, it follows that $X - \{n\} \subsetneq n$, yielding our contradiction.

Define $h : n+1 - \{i\} \rightleftarrows n$, as follows:

$$h(k) = \begin{cases} k & \text{if } k \neq n \\ i & \text{if } k = n \end{cases}$$

(h condenses the "gappy set" $n+1-\{i\}$ to n by placing n into the "gap" created by the removal of i.)

Now, define $g : n \hookrightarrow X - \{n\}$ by $f \circ h^{-1}$. Since $i \notin \operatorname{ran}(h)$ we have $f(i) = n \notin \operatorname{ran}(g)$. But, $X - \{n\} \subsetneq n$, so g is a nontrivial injection on n. $\lightning$

✓ Therefore, there are no nontrivial injections on any natural number.

Corollary

Theorem

The following hold.

1. $n \leq m$ *if and only* $n \preccurlyeq m$.
2. *For each finite set A there is exactly one natural number* $n \approx A$.
3. ω *is infinite.*

Proof of corollary

Proof.

① $n < m$ implies $n \subsetneq m$, so that $n \preccurlyeq m$.
Conversely, $n \preccurlyeq m$ implies $m \not\preccurlyeq n$ by PHP (since $m < n$ implies $m \subsetneq n$), so that $n \leq m$.

② $n \approx A \approx m$ implies $n \approx m$. Thus, $n = m$ by PHP.

③ The map $n \mapsto n + 1$ is a nontrivial injection $\omega \hookrightarrow \omega - \{0\}$. So, ω cannot by finite by PHP.

□

Theorem

Theorem

If X is infinite then $n \prec X$ for all natural numbers n.

Proof.

By induction on n. Note that $0 \preccurlyeq A$ for all sets A.

Suppose $n \preccurlyeq X$ (i.h.) Let $f : n \hookrightarrow X$. Since X is infinite, f is not onto. Let $a \in X - \text{ran}(f)$. Define $g : n + 1 \hookrightarrow X$ by $f \cup \{(n, a)\}$. □

☠ Warning. The theorem does NOT say that $\omega \preccurlyeq X$: It builds a DIFFERENT function for each n, not one single function with domain ω.

Closure conditions of finite sets

Theorem

1. *X is finite and $Y \preccurlyeq X$ implies Y is finite. (H+J, 4.2.4)*
2. *X is finite and f a function implies $f[X] \preccurlyeq X$. (H+J, 4.2.5)*
3. *X, Y is finite implies $X \cup Y, X \times Y, {}^{Y}X, \mathcal{P}(X)$ are all finite. (H+J, 4.2.6, 4.2.8, Exercise 4.2.2)*
4. *If S is a finite family of finite sets then $\bigcup S$ is finite. (H+J, 4.2.7)*

☞ These conditions show that we need the Axiom of Infinity. Axioms 0 to 8 (without Infinity) are all true if we take our universe V to be only the finite sets:

$$H_0 = \emptyset \quad H_{n+1} = \mathcal{P}(H_n) \quad H = \bigcup_{n<\omega} H_n.$$

H is the hereditarily finite sets. I suppose $V = H$ is the finitist thesis.

Proof of ①

① WLOG we may assume $X = n$ is a natural number and $Y \subseteq n$. We will prove that there is an $m \leq n$ such that $Y \approx m$. The proof is by induction on n. The case of 0 is clear.

☞ (i.h.) if $Z \subseteq n$ then there is an $m \leq n$ such that $Z \approx m$.
Let $Y \subseteq n + 1$. Two cases: $n \in Y$ or $n \notin Y$.

- $n \notin Y$. Then $Y \subseteq n$, so there is an $m \leq n < n + 1$ such that $Y \approx m$.
- $n \in Y$. Then $Y - \{n\} \subseteq n$. By (i.h.) there is an $m \leq n$ and function $f : m \rightleftarrows Y - \{n\}$.
  Let $g = f \cup \{(m, n)\}$; then $g : m + 1 \rightleftarrows Y$ and $m + 1 \leq n + 1$.

✓ Therefore, if X is finite and $Y \preccurlyeq X$ then Y is finite.

Countable sets

Definition

A set A is countable if $A \preccurlyeq \omega$; A is countably infinite if A is countable and infinite.

Note on Terminology. Hrbacek and Jech use “countable" to mean “countable and infinite". I will always be explicit when I mean to single-out a countable and infinite set.

Countably infinite sets

All countably infinite sets are equipollent to ω.

Theorem

If $A \preccurlyeq \omega$ and A is infinite then $A \approx \omega$.

Corollary

A set is countable if and only if it is either finite or equipollent to ω.

Proof of theorem

Suppose $A \preccurlyeq \omega$ via $f : A \hookrightarrow \omega$ and A is infinite.
Define a map $g : \omega \hookrightarrow A$ by primitive recursion.

- Let $g(0) = a$ where $f(a)$ is least in the range of f.
- Assume $g(n)$ has been defined. Define $g(n+1) = b$ where $f(b) \in \omega$ is least with $b \in A - \{g(0), \ldots, g(n)\}$.

  This is always possible: since A is infinite and $\operatorname{ran}(g \restriction n+1)$ is finite, $A - \operatorname{ran}(g \restriction n+1)$ is nonempty.

The map $g : \omega \hookrightarrow A$ by “our construction" of g (formally, this is proved by induction; informally, note that we choose a fresh element of A each time in the construction of g.) Thus, $\omega \preccurlyeq A$ and by hypothesis $A \preccurlyeq \omega$, so that by Schröder-Bernstein we have $A \approx \omega$.

Comment on theorem and proof

☞ Why is the hypothesis $A \preccurlyeq \omega$ in the Theorem? It looks like the same argument could be given for the simpler:

> *If A is infinite then* $\omega \preccurlyeq A$*.*

☞ Here is how we construct g:

- ➠ Let $g(0) = a$ where a is any element of A.
- ➠ Assume $g(n)$ has been defined. Define $g(n+1) = b$ where we choose any $b \notin A - \text{ran}(g \restriction n+1)$. (This set is nonempty.)

☞ How do we justify the “arbitrary choice" we continually make at each step? The Axiom of Choice is necessary for justifying making all these “arbitrary choices". (Notice, in our proof we kept “choosing" an element that we could define: the $b \in A - \text{ran}(g \restriction n+1)$ such that $f(b)$ is least. This definition uses the well-ordering of ω.)

Coding sets in ω

☞ Many cardinality arguments involve “coding" one set into another in a 1-1 manner.

We will use the prime numbers for convenience: let p_n denote the nth prime. So, $p_0 = 2, p_1 = 3, p_2 = 5$ etc.

☞ We will also use the following family of mutually disjoint and countably infinite sets for our coding:

$$P_0 = \{2^k \mid k \in \omega\}, \quad P_1 = \{3^k \mid k \in \omega\}, \ldots, \quad P_n = \{p_n^k \mid k \in \omega\}, \ldots$$

Closure under countable sets

Recall, $\mathcal{F}$ is a finite family of countable sets if there is an $n \in \omega$ such that $\mathcal{F} = \{A_i \mid i < n\}$ and each A_i is a countable set.

Theorem

Let A, B be countable sets and $\mathcal{F}$ be a finite family of countable sets.

(a) $A \cup B$ *is countable.*

(b) $\bigcup \mathcal{F}$ *is countable.*

(c) $A \times B$ *is countable.*

(d) $\prod \mathcal{F}$ *is countable.*

(e) *For each $n < \omega$, A^n is countable. (A^n is the set of all sequences of elements of A of length n.)*

(f) $A^{<\omega} = \bigcup_n A^n$ *is countable. ($A^{<\omega}$ is the set of all finite sequences of elements of A.)*

Proof of closure conditions

Let $f : A \hookrightarrow \omega$ and $g : B \hookrightarrow \omega$.

(a). Let $h : A \cup B \hookrightarrow \omega$ be defined by

$$h(x) = \begin{cases} 2^{f(x)+1} & \text{if } x \in A \\ 3^{g(x)+1} & \text{if } x \in B - A \end{cases}$$

(b). The proof is by induction on n, the number of sets in the finite family $\mathcal{F}$ of countable sets.

☞ Basis. When $\mathcal{F} = \emptyset$ (a finite family of finite sets) then $\bigcup \mathcal{F} = \emptyset$ is countable.

☞ Inductive. Suppose (i.h.) that the union of any family of n countable sets is also countable. Let $\mathcal{F}$ be a family of $n + 1$ countable sets.

$\mathcal{F} = \mathcal{F}' \cup \{A_n\}$, where $\mathcal{F}'$ is a family of n countable sets, and so $\bigcup \mathcal{F}'$ is countable (i.h.)

Apply **(a)**: $\bigcup \mathcal{F} = \bigcup \mathcal{F}' \cup A_n$ is countable.

Proof of closure conditions

Let $f : A \hookrightarrow \omega$ and $g : B \hookrightarrow \omega$.

(c). Let $h : A \times B \hookrightarrow \omega$ be defined by

$$h(a, b) = 2^{f(a)+1}3^{g(b)+1}$$

(d). Proof by induction on the size of the family; similar to **(b)**.

(e). Follows from **(d)** since $A^n \approx \prod_{i<n} A$. Here is an explicit construction: let $f : A \hookrightarrow \omega$, now define $h_n : A^n \hookrightarrow \omega$ by

$$h_n(\langle a_i \mid i < n\rangle) = 2^{f(a_0)+1}3^{f(a_1)+1} \dots p_{n-1}^{f(a_{n-1}+1)}.$$

Notice that if $n \neq m$ then A^n and A^m are disjoint as are $\text{ran}(h_n)$ and $\text{ran}(h_m)$.

(f). Let $\{h_n : A^n \hookrightarrow \omega \mid n < \omega\}$ be injections as defined in **(e)**. Define $h : A^{<\omega} \hookrightarrow \omega$ by $h(s) = h_n(s)$ where $s \in A^n$.

Countable ordinals closed under ordinal operations

☞ It follows from parts (a), (c) and (f) of the previous theorem, all our constructions of ordinals numbers using addition, multiplication and exponentiation produced only countable ordinals. (Examples using $\omega + \omega$, $\omega \cdot \omega$ and ω^ω can be found in Lecture 15, slide 10.)

(i) $\alpha + \beta \preccurlyeq \alpha \times \{0\} \cup \beta \times \{1\}$. (Lecture 14, slide 27.)

(ii) $\alpha \cdot \beta \preccurlyeq \beta \times \alpha$. (Lecture 14, slide 27 and Homework 4, Problem 1.)

(iii) Let $\gamma = \max\{\alpha, \beta\}$. Then $\alpha^\beta \preccurlyeq \gamma^{<\omega}$ (where $\gamma^{<\omega}$ is the finite sequences of elements of γ. This holds by Homework 4, Problem 10.)

☞ So, we can never generate uncountable ordinals using the operations of addition, multiplication or exponentiation. We will shortly see a new operation due to Hartogs which will produce uncountable ordinals.

Countable unions of countable sets

☞ Consider the following “theorem":

The countable union of countable sets is countable.

Here is an “argument" for the theorem: Let $\{A_n \mid n < \omega\}$ be a countable family of disjoint countable sets.

Choose an embedding for each n, $f_n : A_n \hookrightarrow \omega$ and code $\bigcup\{A_n \mid n < \omega\}$ by:

$$a \mapsto p_n^{f_n(a)} \qquad \text{where } a \in A_n.$$

Problem. How do we justify the “arbitrary choices" of the maps f_n?

☞ We proved that if A is countable, then so is $A^{<\omega} = \bigcup_n A^n$. But, in this case we could define a family of embeddings: Let $A = \langle a_0, a_n, \ldots\rangle$ be an embedding of A into ω; then the family of embeddings $\{g_n \mid n < \omega\}$ where $g_n : A^n \to \omega$ is defined by $\langle a_{k_0}, \ldots, a_{k_{n-1}}\rangle \mapsto p_0^{k_0} p_1^{k_1} \ldots p_{n-1}^{k_{n-1}}$.

Problems with countable sets

☞ We have cannot prove the following “reasonable" propositions:

(a) If A is infinite then $\omega \preccurlyeq A$.

(b) A countable union of countable sets is countable. In fact, we cannot even prove that $\bigcup_{n<\omega} A_n$ is countable when $A_n \preccurlyeq 2$!!

(c) A countable cartesian product of countable sets is countable.

☞ In each case we are blocked in our proof because we need to be able to make (possibly infinitely many) “arbitrary choices" in our construction of the necessary embedding. In those cases where we could explicitly define each choice, we could carry-out the proof of countability.

☞ The Axiom of Choice baldly states there are always explicitly defined choices. For example, in case (b) it will guarantee that for each countable family $\{A_n \mid n < \omega\}$ there is a countable of embeddings $\{f_n : A_n \hookrightarrow \omega \mid n < \omega\}$.