Math 582
Intro to Set Theory
Lecture 24

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Introduction

☞ This lecture defines the notions same size (equinumerous) and at least in large in size.

☞ Size is measured by the “number of elements", where all properties are stripped from the elements of the set except their distinctness.

☞ The central result is the Schröder-Bernstein Theorem (or Schröder-Bernstein-Cantor Theorem) which simplifies the task of comparing two sets for size.

☞ There are several examples to introduce techniques for comparing the size of sets, and to begin to see patterns for developing an arithmetic based upon the size of sets.

☞ These lectures correspond to H+J Section 4.1 and Section 5.1.

Equinumerous

Definition

$X \approx Y$ iff there is a function $X \rightleftarrows Y$.
$X \preccurlyeq Y$ iff there is a function $X \hookrightarrow Y$.

☛ When $X \approx Y$ we will say that the sets are equinumerous, or equipotent (H+J), or have the same cardinality or size.

☛ When $X \preccurlyeq Y$ we will say the set X is less than or equal to Y in cardinality or size.

Note. H+J write $|A| = |B|$ when they mean $X \approx Y$ and $|A| \leq |B|$ when they mean $X \preccurlyeq Y$. In Chapters 4 and 5, $|X|$ does not denote any set, and will not in H+J until chapter 7 when $|X|$ will denote the cardinal number of X.

Simple Examples

- The only set equinumerous with $\emptyset$ is $\emptyset$. But, $\emptyset \preccurlyeq X$ for every set X.
- The sets $3 = \{0, 1, 2\}$ and $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$ are equinumerous with the correspondence f given by: $0 \mapsto \emptyset \quad 1 \mapsto \{\emptyset\} \quad 2 \mapsto \{\{\emptyset\}\}$

The following are believable, but some will require a fair amount of effort to show. ☺

- $n \preccurlyeq m$ iff $n \leq m$ for every $n, m \in \omega$.
- $n \preccurlyeq \omega$ and $\omega \not\preccurlyeq n$ for every finite ordinal n.
- $\mathbb{N} \preccurlyeq \mathbb{Z}$ and $\mathbb{Q} \preccurlyeq \mathbb{R}$. by the trivial embedding. (That is, we have $\mathbb{N} \subseteq \mathbb{Z}$ and $\mathbb{Q} \subseteq \mathbb{R}$.)
- $\mathbb{N} \approx \mathbb{Z} \approx \mathbb{Q}$ by well-known mappings.
- $\mathbb{R} \not\preccurlyeq \mathbb{Q}$ by Cantor's diagonal argument.

Relations

Lemma

① $\preccurlyeq$ *is transitive and reflexive.*

② $X \subseteq Y$ *implies* $X \preccurlyeq Y$.

③ $\approx$ *is an equivalence relation.*

Proof.

① and ③ are by composing maps. ② is by the identity function on X (the “natural embedding" of X into Y.) □

Schröder-Bernstein Theorem

☞ We proved the Schröder-Bernstein Theorem in Lecture 7, and you can verify that we can carry-out the proof formally in set theory. HW6 will have another version of the proof due to Zermelo.

Theorem (Schröder-Bernstein Theorem)

$A \approx B$ *iff* $A \preccurlyeq B$ *and* $B \preccurlyeq A$.

☞ We can now unambigously define less than in size:

Definition

$X \prec Y$ iff $X \preccurlyeq Y$ and $Y \not\preccurlyeq X$.

Equivalently (by SBT), $X \hookrightarrow Y$ but **NOT**-($X \rightleftarrows Y$).

Cardinal number and Schröder-Bernstein Theorem

☞ Informally, we will write

$$|A| \leq |B| \quad \text{to mean} \quad A \preccurlyeq B$$

and

$$|A| = |B| \quad \text{to mean} \quad A \approx B$$

☞ The Schröder-Bernstein Theorem then says that

$$|A| \leq |B| \;\leftrightarrow\; |A| \leq |B| \,\wedge\, |B| \leq |A|$$

☞ Next week we will produce a representative set for $|A|$, a cardinal number.

Exercises in comparing size

Here are some simple exercises to test your comprehension:

- ✏ When $A \cap B = \emptyset = C \cap D$: $A \preccurlyeq C$ and $B \preccurlyeq D$ implies $A \cup B \preccurlyeq C \cup D$.
- ✏ $A \preccurlyeq C$ and $B \preccurlyeq D$ implies $A \times B \preccurlyeq C \times D$.
- ✏ $1 \times A \approx A$ for all sets A.
- ✏ When $A_0 \approx A_1$ and $A_0 \cap A_1 = \emptyset$: $\quad A_0 \cup A_1 \approx 2 \times A_0$.
  ☞ Generalize to any finite number.

☞ Disjoint union behaves like addition of sizes: when $A \cap B = \emptyset$ we will define

$$|A| + |B| = |A \cup B|.$$

☞ Cartesian product behaves like multiplication of sizes: we will define

$$|A| \cdot |B| = |A \times B|.$$

Cantor's Theorem

Theorem (Cantor's Theorem)

$A \prec \mathcal{P}(A)$ for every set A.

(See H+J Theorem 4.6.2 and Theorem 5.1.8.)

Proof Idea.
$A \preccurlyeq \mathcal{P}(A)$: by $x \mapsto \{x\}$.

$\mathcal{P}(A) \not\preccurlyeq A$: define the Cantor diagonal set for any function $h : A \to \mathcal{P}(A)$ by

$$D_h := \{x \in A \,|\, x \notin h(x).\}$$

Then $D_h \notin \operatorname{ran}(h)$ (✍ verify.)
So, it is not possible that $h : A \twoheadrightarrow \mathcal{P}(A)$

Powerset and Exponentiation

Lemma

${}^A2 \approx \mathcal{P}(A)$ for every set A.

So, $A \prec {}^A2$ for every set A.

Proof.

Associate each $B \subseteq A$ by its characteristic function

$$\chi_B(x) = \begin{cases} 1 & x \in B \\ 0 & x \notin B \end{cases}$$

□

More Exercises in comparing size

Here are some simple exercises to test your comprehension:

- $A \preccurlyeq C$ and $B \preccurlyeq D$ implies ${}^{A}B \preccurlyeq {}^{C}D$.
- $2 \preccurlyeq C$ implies $\mathcal{A} \prec {}^{A}C$.
- $A \approx {}^{1}A$
- $A \times A \approx {}^{2}A$.
  - Generalize to any finite number.

☞ Function spaces behave like exponentiation: we will define

$$|A|^{|B|} = \left|{}^{B}A\right|.$$

So, the last example show $|A|^{|2|} = \left|{}^{2}A\right| = |A \times A| = |A| \cdot |A|$.

Function spaces like exponentiation

Lemma

For all A, B, C the following hold:

(i) ${}^{C}\left({}^{B}A\right) \approx {}^{C\times B}A$.

(ii) ${}^{B\cup C}A \approx {}^{B}A \times {}^{C}A$ *when* $B \cap C = \emptyset$.

(iii) ${}^{A}(B \times C) \approx {}^{A}B \times {}^{A}C$

Proof.

(i). Define $\Phi : {}^{C}\left({}^{B}A\right) \rightleftarrows {}^{C\times B}A$ by $\Phi(f)(c,b) = \left(f(c)\right)(b)$ (Note that $f(c) \in {}^{B}A$.)

(ii). Define $\Psi : {}^{B\cup C}A \rightleftarrows {}^{B}A \times {}^{C}A$ by $\Psi(f) = \left(f \restriction B, f \restriction C\right)$.

(iii). Define $\Gamma : {}^{A}(B \times C) \rightleftarrows {}^{A}B \times {}^{A}C$ by $\Gamma(f)(a) = \left(\text{first}(f(a)), \text{second}(f(a))\right)$, where $\text{first}(x,y) = x$ and $\text{second}(x,y) = y$.

□