Math 582
Intro to Set Theory
Lecture 23

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Axiom 8: Power Set Axiom

Axiom 8: Power Set Axiom:

$$\forall x \exists y \forall z \left(z \subseteq x \ \rightarrow \ z \in y \right)$$

By applying Comprehension $\{z \,|\, z \subseteq x\}$ exists.

Definition

The power set of a set x is the set containing the subsets of x, $\mathcal{P}(x) = \{z \,|\, z \subseteq x\}$.

- The Power set axiom was one of Zermelo's original axioms.
- Power set axiom does not follow from the other axioms we have seen so far: we cannot prove that there is an uncountable set from Axioms 0-7.
- The Power Set Axiom is necessary for defining $\mathbb{R}$, so needed for mathematics.

Function Spaces

We can now define function spaces. Note that if $f : A \to B$ then $f \subseteq A \times B$, so $f \in \mathcal{P}(A \times B)$.

Definition

B^A (or ${}^A B$) is the set of functions f with $\mathrm{dom}(f) = A$ and $\mathrm{ran}(f) \subseteq B$.

Justification. $B^A \subseteq \mathcal{P}(\mathcal{P}(A \times B))$; now, use Comprehension.

- B^A is the standard terminology in mathematics (and used in H+J).
- ${}^A B$ is occasionally used, especially when A, B are ordinals (and ambiguity can ensue.) For example, ${}^3 2$ is a set with 8 functions, but 2^3 is the ordinal 8.

Sequences

Definition

- For $\alpha \in \mathbf{ON}$ an α-sequence is a function s with domain α. We write s_ξ for $s(\xi)$ when $\xi \in \alpha$.
- $A^{<\alpha} = \bigcup_{\xi<\alpha} A^\xi$. (Sometimes written ${}^{<\alpha}A$.)

- "Infinite sequences" in calculus are really functions $s : \omega \to \mathbb{R}$ (where we view s_n as $s(n)$.)
- If we view A as an alphabet (of "letters") then $A^{<\alpha}$ is the set of finite strings formed from A. For example, $2^{<\omega}$ is the set of finite bit strings commonly used in theoretical computer science.

General Products

Definition

☞ Let $S = \langle S_i \mid i \in I \rangle$ (the tuple notation for function used in H+J) be a function with domain I and such that $S(i) = S_i$ for some set S_i. We call such a function S an indexed system of sets.

☞ The product of an indexed system S is the set

$$\prod S = \{f : I \to \bigcup_{i \in I} S_i \mid f(i) \in S_i \text{ for each } i \in I\}$$

When we want to make the indexed set I explicit, we write

$$\prod_{i \in I} S_i \qquad \prod \langle S_i \mid i \in I \rangle.$$

Justification. $\prod S \subseteq \mathcal{P}(I \times \bigcup_{i \in I} S_i)$, so that $\prod S \in \mathcal{P}\Big(\mathcal{P}(I \times \bigcup_{i \in I})\Big)$.

General unions and intersections

- Indexed system of sets S are commonly used with $\bigcup, \bigcap$:

$$\bigcup_{i \in I} S_i \quad \text{meaning} \quad \bigcup \{S_i \mid i \in I\}$$

$$\bigcap_{i \in I} S_i \quad \text{meaning} \quad \bigcap \{S_i \mid i \in I\}$$

- For example, in Lecture 3 (slide 27), we introduced countable intersections and unions:

$$\bigcup_{n}^{\infty} A_n \quad \text{written now as} \quad \bigcup_{n \in \omega} A_n$$

$$\bigcap_{n}^{\infty} A_n \quad \text{written now as} \quad \bigcap_{n \in \omega} A_n$$

Cartesian Products

We did not need the Axiom of Replacement to justify cartesian products $A \times B$. (This is important to know historically, since cartesian products were available to Zermelo and early set theorists before Fraenkel introduced the Replacement Axiom.)

☞ Recall, our official definition of ordered pair, (a, b) as $\{\{a\}, \{a, b\}\}$.
So, $\{a\}, \{a, b\} \in \mathcal{P}(A \cup B)$ and $(a, b) \in \mathcal{P}\big(\mathcal{P}(A \cup B)\big)$;
and by Comprehension

$$A \times B = \{(a, b) \in \mathcal{P}\big(\mathcal{P}(A \cup B)\big) \,\big|\, a \in A \ \wedge \ b \in B\}$$

Countable vs. Uncountable sets

Definition

A set x is countable if there is a $f : x \xrightarrow{1\text{-}1} \omega$. A set is uncountable if there is no such 1-1 function mapping x into ω.

- $\omega + \omega$ is countable: map ω into the even numbers and $\{\omega + n \,|\, n < \omega\}$ into the odd numbers.
- $\omega \cdot \omega$ is countable: Let $p_0, p_1, p_2, \ldots$ be a list of the prime numbers, and map $(m, n) \mapsto p_m^n$. (Recall, $\omega \cdot \omega$ is isomorphic to $\omega \times \omega$ with lexicographic order, from Homework 5.)
- ω^ω is countable: by Exercise 8 from Homework 5 we can think of ω^ω as consisting of functions $f : \omega \to \omega$ with finite domain. If $f \in \omega^\omega$ then map $f \mapsto \prod_{i \in \mathrm{dom}(f)} p_i^{f(i)}$. ($\prod$ here is multiplication on ω, not a set of functions ☺.)

$\mathcal{P}(\omega)$ is uncountable

☯ Shades of Russell. Actually, Russell's paradox (1903) is based on this argument of Cantor's (1891). The ♥ is a diagonal argument.

Theorem

$\mathcal{P}(\omega)$ is uncountable.

Proof.

☞ Suppose $h : \mathcal{P}(\omega) \xrightarrow{1\text{-}1} \omega$. Define

$$\mathfrak{r} = \{n \mid \exists x\, (n = h(x) \ \wedge\ n \notin x)\}$$

(this makes sense since h is 1-1 and $x \subseteq \omega$.)

☞ Let $n = h(\mathfrak{r})$. Either $n \in \mathfrak{r}$ or $n \notin \mathfrak{r}$.

✂ If $n \in \mathfrak{r}$ then since $n = h(\mathfrak{r})$, $n \notin \mathfrak{r}$.

✂ So, $n \notin \mathfrak{r}$. But, since $n = h(\mathfrak{r})$, $n \in \mathfrak{r}$. ↯

✓ Therefore, there can be no 1-1 function from $\mathcal{P}(\omega)$ into ω.

□