Math 582
Intro to Set Theory
Lecture 22

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Ordinal number as order type

Cantor, *Contributions to the Founding of the Theory of Transfinite Number* (1895), §7:

> *Every ordered set U has a definite 'order type', . . . which we will denote by $\overline{U}$. By this we understand the general concept which results from U if we only abstract from the nature of the elements u, and retain the order or precedence among them. Thus the ordinal type $\overline{U}$ is itself an ordered set whose elements are units which have the same order of precedence amongst one another as the corresponding elements of U, from which they are derived by abstraction. . . . A simple consideration shows that two ordered sets have the same ordinal type if and only if they are similar, so that of the two formulas $U =_o V$ and $\overline{U} = \overline{V}$, one is always a consequence of another.*

Ordinal number as order type

- Cantor was speaking about arbitrary *totally ordered sets*, although we will only consider the order types of *well-ordered sets*.
- Cantor uses *similar* where we would say *order isomorphic*. He writes $U =_o V$ where we would write $U \cong V$, where U, V are ordered sets.
- Cantor had shown that any two well-ordered sets are comparable: where $U \leq_o V$ if there is a $f : U \hookrightarrow V$ which is order preserving $(x \leq y \rightarrow f(x) \leq y.)$
- Cantor isolates two properties the order type should satisfy (and I have added a natural third condition implied by his "simple consideration"):
  ① $U =_o \overline{U}$,
  ② $U =_o V \quad \leftrightarrow \quad \overline{U} = \overline{V}$,
  ③ $U \leq_o V \quad \leftrightarrow \quad \overline{U} \leq \overline{V}$,
- Informally, Cantor's notion of "abstraction" corresponds to our forming "equivalence classes" of well-ordered sets, $[U] = \{V \mid U =_o V\}$, and using these as *order types*. Formally, we follow von Neumann (1920s) and take the *ordinal numbers* as the *order types* of well-ordered sets.

Goals

The main result (compare: H+J, Theorem 6.3.1) is

> ♥ *For each well-ordered set (A, R) there is a unique ordinal α such that $(A, R) \cong (\alpha, \in)$. We write type$(A) = \alpha$.*

It then follows as a consequence that

- ☛ $U \leq_o V \quad \leftrightarrow \quad \text{type}(U) \leq \text{type}(V)$. (So, any two well-ordered sets are comparable.)
- ☛ We can define functions by transfinite recursion on any well-ordered set.

The second result we will take from ♥ is to define an equivalent combinatorial definition of addition and multiplication on the ordinals (due to Cantor), which is "more natural" than our official definition by transfinite recursion.

First, we learn to count . . .

Let (A, R) be a well-ordered set. Let's count how many elements are in A:

- ✍ If A is nonempty, then A has a least element a_0.
- ✍ If $A \neq \{a_0\}$, then $A - \{a_0\}$ has a least element, a_1.
- ✍ If $A \neq \{a_0, a_1\}$, then $A - \{a_0, a_1\}$ has a least element, a_2.
- ✍ . . . (etc. selecting a_n for $n < \omega$, if possible.)
- ✍ If A is infinite, it will have an ω-sequence of elements, $\{a_0, a_1, a_2, \ldots\}$. If this does not exhaust A, then $A - \{a_n \mid n < \omega\}$ has a least element, a_ω.
- ✍ . . . (etc., selecting a_ξ for ordinals ξ, if possible.)
- ✍ When we have exhausted A we will have a 1-1 correspondence between A and an initial segment of **ON**, α. So, let type$(A) = \alpha$.

☕ The process must exhaust A before **ON** since A is a set and **ON** is a proper class – by the Replacement Axiom.

Carrying-out the counting strategy

Let 🛑 be any set which is not a member of A. Define
$\mathbf{T} : \mathbf{ON} \to A \cup \{🛑\}$ by

(a) $\mathbf{T}(\xi) =$ 🛑 if $A - \text{ran}(\mathbf{T} \restriction \xi) = \emptyset$ (we have exhausted A), or

(b) $\mathbf{T}(\xi) = a$ where a is R-least in $A - \text{ran}(\mathbf{T} \restriction \xi)$ (count the next element in A.)

☞ Let type(A) be the least ordinal α such that $\mathbf{T}(\alpha) =$ 🛑. The proof would show this works.

☕ It is more convenient, and (perhaps) more natural, to define our map $\mathbf{T} : A \to \mathbf{ON}$. The trouble is that in order to count A we need to define $\mathbf{T}$ by transfinite recursion on A, which we haven't proven yet.

Uniqueness

Lemma

If $f : \alpha \rightleftarrows \beta$ is an isomorphism from $(\alpha, <)$ to $(\beta, <)$, then f is the identity map, and hence $\alpha = \beta$.

☞ The lemma implies that $(\alpha, <)$ has no automorphisms besides the identity map. This is in contrast with arbitrary linearly ordered sets which can have nontrivial automorphisms.

For example, $\mathbb{Q}$ has lots of nontrivial automorphisms: for any positive rational $c \neq 1$

$$q \mapsto cq$$

is a nontrivial automorphism.

Proof of Uniqueness

Proof.

Let $f : \alpha \rightleftarrows$ 1-1,ontoβ be an order-preserving map. Fix $\xi \in \alpha$; then,

$$f(\xi) \stackrel{1}{=} \{\nu \in \beta \,|\, \nu < f(\xi)\} \stackrel{2}{=} \{f(\mu) \,|\, \mu < \xi\} \qquad ✺$$

since (1) $f(\xi)$ is an ordinal and (2) f is order-preserving.

We prove that $f(\xi) = \xi$ by transfinite induction on ξ. Suppose (i.h.) for all $\mu < \xi$ that $f(\mu) = \mu$. Then

$$\begin{aligned} f(\xi) &= \{f(\mu) \,|\, \mu < \xi\} && \text{by ✺} \\ &= \{\mu \,|\, \mu < \xi\} && \text{i.h.} \\ &= \xi \end{aligned}$$

□

Initial segments of ordered sets

Let (B, R) be an ordered set. Then

- ✍ Define $B[a] = \{bRa \mid b \in B\}$ – the segment of B below a.
- ✍ $A \subset B$ is an initial segment of B if for each $a \in A$, $B[a] \subset A$. (You can think of our requirement that an ordinal is a transitive set as requiring that it is an initial segment of **ON**.)

Lemma

Let (B, R) and (C, S) be ordered sets and $h : B \to C$ an order isomorphism. If U is an initial segment of B then $h[U]$ is an initial segment of C.

Initial segments of ordered sets

Proof.

Let $h : B \to C$ be an isomorphism between ordered sets (B, R) and (C, S). Let U be an initial segment of B.

☞ Suppose $d \in h[U]$ and cSd; show $c \in h[U]$.

Let $x, y \in B$ with $h(x) = c\,S\,d = h(y)$; so, xRy since h is order-preserving.

$d \in h[U]$ implies $y \in U$; but, xRy and U is an initial segment, so $x \in U$.

Thus $c = h(x) \in h[U]$.

□

Initial segments of well-ordered sets

If (W, R) is a well-ordered set then each initial segment of W is actually determined by an element of W:

Lemma

If W is a well-ordered set and $A \subsetneq W$ is an initial segment of W, then there is an $a \in W$ such that $W[a] = A$ (where $W[a] = \{x \in W \mid xRa\}$.)

Note 1. We proved every initial segment of **ON** is an ordinal; but the proof there depended upon the definition of ordinal.

Note 2. The Lemma need not hold on nonwellfounded sets. For example,

$$\mathbb{Q}[\sqrt{2}] = \{q \in \mathbb{Q} \mid q < \sqrt{2}\}$$

is an initial segment, but not determined by any rational number.

Proof of Lemma

Proof.

Let W be a well-ordered set and $A \subsetneq W$ an initial segment of W.
Since $W - A \neq \emptyset$, this set has an R-least element a.
We show $A = W[a]$.

☞ $W[a] \subseteq A$: if xRa then since a is R-least it follows that $x \in A$.

☞ $A \subseteq W[a]$: Let $x \in A$. By linearity, one of aRx, $a = x$ or xRa.
- ➙ $a = x$: impossible, since $a \notin A$.
- ➙ aRx: impossible, since $x \in A$ and A is an initial segment of W.

So, xRa and thus, $x \in W[a]$.

✓ Therefore, $A = W[a]$.

□

Order types of well-ordered sets

Theorem

For every well-ordered set $(W, \prec)$*, there is a unique ordinal* α *such that* $(W, \prec) \cong (\alpha, \in)$.

Definition

Let $(W, \prec)$ by a well-ordered set. The order type of $(W, \prec)$ is the unique ordinal α isomorphic to $(W, \prec)$. We write $\text{type}(W, \prec) = \alpha$, or simply $\text{type}(W) = \alpha$ where the relation $\prec$ is understood from the context.

Proof

☞ Let 🛑 $\notin W$.
(See Lecture 9, slide 12 – the proof that there is no universal set.)

☞ Define the class function $\mathbf{G} : \mathbf{V} \to \mathbf{V}$ on a set x by

➠ If $W - \text{ran}(x) \neq \emptyset$, then $\mathbf{G}(x)$ is the $\prec$-least member of $W - \text{ran}(x)$; otherwise, $\mathbf{G}(x) =$ 🛑.

☞ By the Transfinite Recursion Theorem there exists a class function $\mathbf{T} : \mathbf{ON} \to W \cup \{$🛑$\}$ which satisfies

(a) $\mathbf{T}(\beta) =$ 🛑 if $W - \text{ran}(\mathbf{T} \restriction \beta) = \emptyset$
(we have exhausted W), or

(b) $\mathbf{T}(\beta) = w$ where w is $\prec$-least in $W - \text{ran}(\mathbf{T} \restriction \beta)$
(count the next element in W.)

Proof – continued

① If $\alpha < \beta$ and $\mathbf{T}(\alpha) =$ 🛑, then $\mathbf{T}(\beta) =$ 🛑.

Proof.

Suppose $\mathbf{T}(\alpha) = \mathbf{G}(\mathbf{T} \upharpoonright \alpha) =$ 🛑, then $W - \text{ran}(\mathbf{T} \upharpoonright \alpha) = \emptyset$.

If $\alpha < \beta$, then $\mathbf{T} \upharpoonright \alpha \subseteq \mathbf{T} \upharpoonright \beta$, and so $W - \text{ran}(\mathbf{T} \upharpoonright \beta) = \emptyset$.

Thus, $\mathbf{T}(\beta) = \mathbf{G}(\mathbf{T} \upharpoonright \beta) =$ 🛑,

So, if $\alpha < \beta$, then $\mathbf{T}(\beta) =$ 🛑. $\square$

Proof – continued

② If $\alpha < \beta$ and $\mathbf{T}(\beta) \neq$ 🛑, then $\mathbf{T}(\alpha) \prec \mathbf{T}(\beta)$.

Proof.

From the hypothesis and ① we have $\mathbf{T}(\alpha) \neq$ 🛑, so that $\mathbf{T}(\alpha)$ is $\prec$-least in $W - \text{ran}(\mathbf{T} \upharpoonright \alpha)$.

Since $\mathbf{T} \upharpoonright \alpha \subseteq \mathbf{T} \upharpoonright \beta$, it follows that $\mathbf{T}(\beta) \in W - \text{ran}(\mathbf{T} \upharpoonright \alpha)$, so that $\mathbf{T}(\alpha) \preceq \mathbf{T}(\beta)$.

On the other hand, $\mathbf{T}(\alpha) \in \text{ran}(\mathbf{T} \upharpoonright \beta)$, but $\mathbf{T}(\beta) \notin \text{ran}(\mathbf{T} \upharpoonright \beta)$. So, $\mathbf{T}(\alpha) \neq \mathbf{T}(\beta)$. $\square$

Proof – continued

③ There is an α such that $\mathbf{T}(\alpha) = $ 🛑.

Proof.

Suppose not. Let $U \subseteq W$ be defined by

$$U = \{w \in W \mid \exists\alpha\, \mathbf{T}(\alpha) = w\}.$$

So, $\mathbf{T} : \mathbf{ON} \rightleftarrows U$ by ②.

Thus, $\mathbf{T}^{-1} : U \to \mathbf{ON}$, and so **ON** is a set by Replacement and Comprehension. ↯

☞ Let α be least with $\mathbf{T}(\alpha) = $ 🛑, and let $f : \alpha \to W$ be the set function given by $f = \mathbf{T} \restriction \alpha$.

f is surjective by ① and is an isomorphism by ②. □

A technical lemma

☞ The following technical lemma will be useful in our characterization theorem to follow.

Lemma

Let (W, R) a well-ordered set.

Suppose $W = \bigcup_{\xi<\gamma} W_\xi$ where W_ξ is an initial segment of W_η whenever $\xi < \eta < \gamma$. Then

$$type(W) = \sup\{type(W_\xi) \mid \xi < \gamma\}$$

Proof of lemma

Proof.

☞ For each $\xi < \gamma$, let $h_\xi : W_\xi \rightleftarrows \text{type}(W_\xi)$ be an isomorphism.
$\{h_\xi \mid \xi < \gamma\}$ is a family of compatible functions:
If $\xi < \eta$, then $h_\xi = h_\eta \restriction W_\xi$, since W_ξ is an initial segment of W_η.

☞ Let $h = \bigcup_{\xi<\gamma} h_\xi$, so that $h : W \to \mathbf{ON}$.
$\text{ran}(h) = \text{type}(W)$:
Since $h \restriction W_\xi = h_\xi$ is an ordinal, $h[W] = \bigcup_\xi h[W_\xi]$ is an ordinal, and is an order isomorphism.

☞ Thus,

$$\text{type}(W) = \text{ran}(h) \stackrel{1}{=} \bigcup_{\xi<\gamma} \text{ran}(h_\xi) \stackrel{2}{=} \sup\{\text{type}(W_\xi) \mid \xi < \gamma\}$$

where (1) always holds for ranges and (2) is by definition of sup.

Characterization of ordinal addition and multipllication

Definition

Let $(W, <)$ and $(V, \prec)$ be well-orders.

☞ Let $W \oplus V$ be the set $(\{0\} \times W) \cup (\{1\} \times V)$ with the ordering $\lhd$:

$(i, x) \lhd (j, y)$ iff $i < j$, or $i = j = 0$ and $x < y$, or $i = j = 1$ and $x \prec y$.

$W \oplus V$ places W before V.

☞ Let $W \otimes V$ be the set $W \times V$ with the ordering $\lhd$:

$(u, v) \lhd (x, y)$ iff either $u < x$, or $u = x$ and $v \prec y$.

$W \otimes V$ replaces every element $w \in W$ with a copy of V. This is the lexicographic ordering.

☞ $W \oplus V$ and $W \otimes V$ are well-ordered by exercises 1 and 2 from HW3 (Week 5).

Characterization of ordinal addition and multipllication

☞ The next theorem provides a combinatorial characterization of ordinal addition and multiplication. This is the way Cantor actually defined these operations. HW 5 provides a combinatorial characterization of ordinal exponentiation (which Cantor defined by transfinite recursion, as we defined it.)

Theorem

Let α, β ordinals.

(a) $\alpha + \beta =$ *type*$(\alpha \oplus \beta)$

(b) $\alpha \cdot \beta =$ *type*$(\beta \otimes \alpha)$

(Compare to H+J, Theorems 5.3 and 5.8. The proof of part (b) is a homework problem.)

Proof of Theorem, part (a)

(a). The proof is by induction on β.

☞ $\beta = 0$. Since $\{1\} \times \emptyset = \emptyset$,

$$\text{type}(\alpha \oplus 0) = \text{type}(\alpha) = \alpha = \alpha + 0.$$

☞ $\beta = \gamma + 1$. Let $W = \alpha \oplus \beta$.
Since $(1, \gamma)$ is the greatest element of W:

$$W[(1, \gamma)] = \alpha \oplus \gamma,$$

so

$$\begin{aligned} \text{type}(W) &= \text{type}(\alpha \oplus \gamma) + 1 \\ &= (\alpha + \gamma) + 1 \qquad \text{i.h.} \\ &= \alpha + \beta \end{aligned}$$

Proof of Theorem, part (a)

☞ β a limit. Note that

$$\begin{aligned} \alpha \oplus \beta &= \alpha \oplus \bigcup_{\gamma<\beta} \gamma \\ &= \bigcup_{\gamma<\beta} (\alpha \oplus \gamma). \end{aligned}$$

Furthermore, $\alpha \oplus \xi$ is an initial segment of $\alpha \oplus \gamma$ when $\xi < \gamma$.
So, by the previous technical lemma,

$$\begin{aligned} \text{type}(\alpha \oplus \beta) &= \sup\{\text{type}(\alpha \oplus \gamma) \mid \gamma < \beta\} \\ &= \sup\{\alpha + \gamma \mid \gamma < \beta\} \qquad \text{i.h.} \\ &= \alpha + \beta \end{aligned}$$