Math 582
Intro to Set Theory
Lecture 21

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Multiplication on **ON**

Definition (Ordinal multiplication)

For all ordinals β,

$$\begin{aligned} \beta \cdot 0 &= 0 \\ \beta \cdot (\alpha + 1) &= \beta \cdot \alpha + \beta \\ \beta \cdot \alpha &= \sup\{\beta \cdot \xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

Ordinal Multiplication and Normality

Theorem

If $\alpha > 0$, then the function $(\xi \mapsto (\alpha + \xi))$ is normal.

Proof.

See Lecture 19, slide 5 for the theorem being applied.

☞ Let $\alpha > 0$. The function is continuous by definition (limit clause). It is enough to check the successor clause is order preserving:

$$\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha > \alpha \cdot \beta,$$

by the Order Lemma (a) for Lecture 19.

□

Order Laws

Theorem (Order Laws)

For all α and β

(a) *If $\alpha \neq 0$, then $\beta \leq \alpha \cdot \beta$.*

(b) *If $\alpha \neq 0$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.*

(c) $\alpha, \beta \neq 0 \ \rightarrow \ \alpha \cdot \beta \neq 0$.

(d) *If $\beta < \gamma$, then $\beta \cdot \alpha \leq \gamma \cdot \alpha$.*

Proof (a) and (b) are immediate by normality.

(c) follows from (a): since $\alpha \neq 0$

$$1 \leq \beta \leq \alpha \cdot \beta$$

(d). By transfinite induction (next slide).

Proof

☞ $\beta < \gamma \;\rightarrow\; \beta \cdot \alpha \leq \gamma \cdot \alpha$.

(d). Proof by Transfinite Induction on α. The case $\alpha = 0$ is trivial, and successor follows from the order properties of addition.

If α is a limit, we assume (c) for $\delta < \alpha$.

$$\begin{aligned} \beta \cdot \alpha &= \sup\{\beta \cdot \delta \,|\, \delta < \alpha\} \\ &\leq \sup\{\gamma \cdot \delta \,|\, \delta < \alpha\} && \text{i.h.} \\ &= \gamma \cdot \alpha. \end{aligned}$$

0,1 Laws

Theorem (0,1 Laws)

For all α and β

(a) $0 \cdot \alpha = \alpha \cdot 0 = 0$.

(b) $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$.

(c) *If $\alpha \neq 0$ and $\beta > 1$, then $\alpha < \alpha \cdot \beta$*

☞ (a) and (b) are easy transfinite inductions. (c) follows immediately from (b) and the order property (b) from the previous slide:

$$\alpha \neq 0 \;\wedge\; 1 < \beta \;\rightarrow\; \alpha \cdot 1 < \alpha \cdot \beta$$

Distributivity and Associativity

Theorem (Associativity)

For all α, β, γ

(a) *(Distributivity)* $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

(b) *(Associativity)* $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

(a). This is an exercise from HW9.

(b). The proof is by transfinite induction on γ. I leave the case of $\gamma = 0$ and successor to you (it is the same as for the natural numbers, and uses Distributivity). I will do the limit case.

Proof

Proof.

- If any of α, β or γ are zero, then the equality is $0 = 0$. Assume none are zero. Note that $\alpha \cdot \beta \neq 0$ as well.
- Let γ be a limit. Define

$$F(\xi) = \alpha \cdot \xi \qquad G(\xi) = \beta \cdot \xi \qquad H(\xi) = (\alpha \cdot \beta) \cdot \xi.$$

Each function is normal. So,

$$\begin{aligned}
\alpha \cdot (\beta \cdot \gamma) &= F(G(\gamma)) \\
&= \sup\{F(G(\delta)) \mid \delta < \gamma\} && F \circ G \text{ normal, (L.20, s. 11)} \\
&= \sup\{\alpha \cdot (\beta \cdot \delta) \mid \delta < \gamma\} \\
&= \sup\{(\alpha \cdot \beta) \cdot \delta \mid \delta < \gamma\} && \text{i.h} \\
&= \sup\{H(\delta) \mid \delta < \gamma\} \\
&= H(\gamma) = (\alpha \cdot \beta) \cdot \gamma && H \text{ normal}
\end{aligned}$$

Division Algorithm

☞ The next theorem, the division algorithm, specializes to the ordinary division algorithm in the case of the natural numbers. It plays a central role in the normal form theorem (see H+J, Theorem 6.6.4).

Theorem

If α and β are given with $\beta \neq 0$, then there exist unique γ and δ such that $\alpha = \beta \cdot \gamma + \delta$ and $\delta < \beta$.

Proof

☞ Let $F_\beta(\gamma) = \beta \cdot \gamma$. Since F_β is normal and $F_\beta(0) = 0 \leq \alpha$, it follows by the Bracket Theorem (L.20, s. 13) that there is a unique γ such that

$$\beta \cdot \gamma \leq \alpha < \beta \cdot (\gamma + 1).$$

☞ Let $G_{\beta \cdot \gamma}(\delta) = \beta \cdot \gamma + \delta$. Since $G_{\beta \cdot \gamma}$ is normal and $G_{\beta \cdot \gamma}(0) = \beta \cdot \gamma \leq \alpha$, it follows by the Bracket Theorem that there is a unique δ such that

$$\beta \cdot \gamma + \delta \leq \alpha < \beta \cdot \gamma + (\delta + 1) = \big(\beta \cdot \gamma + \delta\big) + 1$$

It follow that $\beta \cdot \gamma + \delta = \alpha$.

Since $\alpha < \beta \cdot (\gamma + 1) = \beta \cdot \gamma + \beta$, it follows that $\delta < \beta$.

This proves existence.

Proof – continued

☞ Suppose γ' and δ' also satisfy the conditions of the Theorem:

$$\alpha = \beta \cdot \gamma' + \delta' \qquad \delta' < \beta.$$

Since $\delta' < \beta$

$$\beta \cdot \gamma' \leq \alpha < \beta \cdot \gamma' + (\delta' + 1) \leq \beta \cdot (\gamma' + 1),$$

so, $\gamma' = \gamma$ by the uniqueness of γ'.

☞ Similarly,

$$\beta \cdot \gamma + \delta' \leq \alpha < \beta \cdot \gamma + (\delta' + 1)$$

which implies that $\delta = \delta'$ by the uniqueness of δ.

Exponentiation on **ON**

Definition (Ordinal exponentiation)

For all ordinals β,

$$\begin{aligned} \beta^0 &= 1 \\ \beta^{\alpha+1} &= \beta^\alpha \cdot \beta \\ \beta^\alpha &= \sup\{\beta^\xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

Ordinal Exponentiation and Normality

Theorem

If $\alpha > 1$, then the function $(\xi \mapsto (\alpha^\xi))$ is normal.

Proof.

See Lecture 19, slide 5 for the theorem being applied.

☞ Let $\alpha > 1$. The function is continuous by definition (limit clause). It is enough to check the successor clause is order preserving:

$$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha > \alpha^\beta,$$

$\alpha^\beta \neq 0$ for all β by easy transfinite induction. The rest follows by the order property for multiplication since $\alpha > 1$. □

0,1 Laws

Theorem (0,1 Laws)

For all α,

(a) $0^\alpha = 0$ *if α is a successor.*

(b) $0^\alpha = 1$ *if $\alpha = 0$ or is a limit ordinal.*

(c) $1^\alpha = 1$.

(d) $\alpha^0 = 1$.

(e) $\alpha^1 = \alpha$.

(f) *If $1 < \alpha, \beta$, then $1 < \alpha^\beta$*

☞ Easy transfinite inductions. Note (a) and (b) can be proven together by transfinite induction. When α is a limit

$$0^\alpha = \sup\{0^\delta \mid \delta < \alpha\} = 1.$$

(f) follows from (e) together with normality (order preserving).

Order Laws

Theorem (Order Laws)

For all α,

(a) *If* $\alpha > 1$, *then* $\beta \leq \alpha^{\beta}$.

(b) *If* $\alpha > 1$ *and* $\beta < \gamma$, *then* $\alpha^{\beta} < \alpha^{\gamma}$.

(c) *If* $\alpha > 1$ *and* $\beta > 1$, *then* $\alpha < \alpha^{\beta}$.

(d) *If* $\beta < \gamma$, *then* $\beta^{\alpha} \leq \gamma^{\alpha}$.

☞ (a) and (b) follow by Normality: let $F_{\alpha}(\xi) = \alpha^{\xi}$. Then, by normality

$$(a) \qquad \beta \leq F_{\alpha}(\beta)$$
$$(b) \qquad \beta < \gamma \ \rightarrow \ F_{\alpha}(\beta) < F_{\alpha}(\gamma).$$

(c) follows from (b).

(d) is an easy induction, as in part (d) on slide 5 for multiplication.

Exponent Laws

Theorem (Exponent Laws)

For all α, β, γ,

(a) $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$.

(b) $(\alpha^{\beta})^{\gamma} = \alpha^{(\beta \cdot \gamma)}$.

☞ Proof by transfinite induction. Homework 9 problems.