Math 582
Intro to Set Theory
Lecture 20

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

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Normal functions defined

Definition

Let $F : \mathbf{ON} \to \mathbf{ON}$.

- F is order preserving if $\forall \alpha, \beta \in \big(\alpha < \beta \;\to\; F(\alpha) < F(\beta)\big)$.
- F is continuous if for every limit ordinal α,
  $F(\alpha) = \sup\{F(\beta) \,|\, \beta < \alpha\}$.
- F is normal if F is order preserving and continuous.

Notes.

- F is a class function, so should, by convention, by boldface. I will use capital F, G, H for normal functions in this lecture. In later lectures I will always make it clear these will be normal functions.
- There is no reason that F could not be a set function with an ordinal $\alpha > \omega$ as domain. It may be convenient to allow this later.

Theorem: Normal is increasing

Theorem

*Let F : **ON** $\to$ **ON** be a normal function. Then $\alpha \leq F(\alpha)$.*

Proof.

By Transfinite induction on α.

Suppose that $\beta \leq F(\beta)$ for all $\beta < \alpha$.
So, $\beta \leq F(\beta) < F(\alpha)$ for every $\beta < \alpha$.

Thus, $\alpha \subseteq F(\alpha)$, equivalently, $\alpha \leq F(\alpha)$.
(Lemma 3 of Lecture 17, Slide 24.) □

Sufficient condition for normality

☞ The following sufficient condition for normality simplifies the checking of order-preserving for normality.

Theorem

Let F be a continuous function such that $F(\beta) < F(\beta + 1)$ for all β. Then F is normal.

Proof. The proof of order-preserving is by Transfinite Induction on β:

$$\forall \alpha, \beta \left(\alpha < \beta \ \to \ F(\alpha) < F(\beta)\right).$$

Proof

$$\forall \alpha, \beta\, (\alpha < \beta \;\rightarrow\; F(\alpha) < F(\beta)).$$

- ☞ $\beta = 0$. The antecedent is false for all α.
- ☞ $\beta = \gamma + 1$. Suppose $\alpha < \beta = \gamma + 1$.
  If $\alpha \leq \gamma$, then

$$F(\alpha) \leq F(\gamma) < F(\gamma + 1) = F(\beta),$$

by the i.h. and by assumption on F.

- ☞ β is a limit. Suppose $\alpha < \beta$. For some δ, $\alpha < \gamma < \beta$. So,

$$\begin{aligned} F(\alpha) &< F(\gamma) && \text{i.h.} \\ &\leq \sup\{F(\xi) \mid \xi < \beta\} && \\ &= F(\beta) && \text{continuity.} \end{aligned}$$

✓ Thus, order-preserving holds, and F is normal.

Example: Ordinal operations

☞ We already have several interesting examples of normal functions.

Theorem

Fix α. Then

1. $(\xi \mapsto (\alpha + \xi))$ *is normal.*
2. $(\xi \mapsto (\alpha \cdot \xi))$ *is normal, provided* $\alpha > 0$.
3. $(\xi \mapsto (\alpha^{\xi}))$ *is normal, provided* $\alpha > 1$.

Proof

Proof.

By the previous Theorem, we need only show $F(\beta) < F(\beta+1)$, since each of the ordinal operators are continuous by definition.

(a). $\alpha + (\beta + 1) = (\alpha + \beta) + 1 > \alpha + \beta$.

(b). Let $\alpha > 0$. Then $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha > \alpha \cdot \beta$, by the Order Lemma (a) for Lecture 19.

(c). Let $\alpha > 1$. Then $\alpha^{\beta+1} = \alpha^{\beta} \cdot \alpha > \alpha^{\beta}$, once we prove $\alpha^{\beta} \neq 0$ for any β and the following Order Lemma for multiplication:

$$\alpha > 0 \ \wedge \ \beta < \gamma \ \rightarrow \ \alpha \cdot \beta < \alpha \cdot \gamma.$$

□

Theorem: limits of normal functions

Theorem

Let F be a normal function and α a limit ordinal.
Then $F(\alpha)$ is also a limit ordinal.

Proof.

Show that $\{\beta \mid \beta < F(\alpha)\}$ has no greatest element.

☞ Suppose $\beta < F(\alpha) = \sup\{F(\gamma) \mid \gamma < \alpha\}$.
By continuity $\beta < F(\gamma)$ for some $\gamma < \alpha$, and by order increasing, $F(\gamma) < F(\alpha)$.

So, if $\beta < F(\alpha)$, there is a $\delta (= F(\gamma))$ with $\beta < \delta < F(\alpha)$.

✓ $F(\alpha)$ has no greatest element. □

Theorem: limiting limits

Lemma

Let F be normal and $\beta < \alpha$. Then

$$\sup\{F(\gamma) \,|\, \gamma < \alpha\} = \sup\{F(\gamma) \,|\, \beta \leq \gamma < \alpha\}.$$

Proof.

If $\delta < \beta$, then $F(\delta) < F(\beta)$, so $F(\delta) \subseteq F(\beta)$. Hence

$$\sup\{F(\gamma) \,|\, \gamma < \alpha\} \subseteq \sup\{F(\gamma) \,|\, \beta \leq \gamma < \alpha\} \subseteq \sup\{F(\gamma) \,|\, \gamma < \alpha\}.$$

✓ Thus, $\sup\{F(\gamma) \,|\, \gamma < \alpha\} = \sup\{F(\gamma) \,|\, \beta \leq \gamma < \alpha\}$. □

Normal functions closed under composition

Theorem

Let F and G be normal functions. Then $G \circ F$ is also normal.

Proof. $G \circ F$ is increasing: for any α and β

$$\alpha < \beta \;\rightarrow\; F(\alpha) < F(\beta) \;\rightarrow\; G(F(\alpha)) < G(F(\beta)).$$

$G \circ F$ is continuous: let γ be a limit ordinal. Show

$$G(F(\gamma)) = \sup\{G(F(\xi)) \,|\, \xi < \gamma\}.$$

Proof – continued

☞ Since γ is a limit and F is normal, so $F(\gamma)$ is also a limit.
G is also normal, so $G(F(\gamma))$ is a limit and

$$G(F(\gamma)) = \sup\{G(\beta) \mid \beta < F(\gamma)\}.$$

☞ Choose $\beta < F(\gamma)$, so for some $\xi < \gamma$:

$$\beta < F(\xi) < F(\gamma), \quad \text{so} \quad G(\beta) < G(F(\xi)) < G(F(\gamma)).$$

Since $\beta < F(\gamma)$ was arbitrary,

$$\begin{aligned} G(F(\gamma)) &= \sup\{G(\beta) \mid \beta < F(\gamma)\} \\ &\leq \sup\{G(F(\xi)) \mid \xi < \gamma\} \\ &\leq G(F(\gamma)). \end{aligned}$$

Theorem: bracketing

☞ I conclude this survey of properties of normal functions with an important “bracketing" condition.
Compare to Lemma 6.6.2, p. 124, of H+J.

Theorem

Let F be a normal function and α an ordinal for which there is a β with $F(\beta) \leq \alpha$. Then there is a unique δ such that $F(\delta) \leq \alpha < F(\delta + 1)$.

Note. The theorem relies on $\text{dom}(F) = \textbf{ON}$ and $\alpha \leq F(\alpha)$. If we take the domain of F to be an ordinal, we need the extra condition that there is some γ with $\alpha \leq F(\gamma)$.

Proof

☞ Let γ be least such that $\alpha < F(\gamma)$ (which must exist, since $\alpha < F(\alpha + 1)$).

☞ $\gamma > 0$, since there is some β with $F(\beta) \leq \alpha$, so $\beta < \gamma$.

☞ γ is not a limit. Otherwise, $\alpha < F(\gamma) = \sup\{F(\xi) \mid \xi < \gamma\}$, so that $\alpha < F(\xi)$ for some $\xi < \varepsilon$.

☞ So, $\gamma = \delta + 1$. Thus, $F(\delta) \leq \alpha < F(\delta + 1)$, proving existence.

☞ Uniqueness follows by order: if $\varepsilon \neq \delta$, then either
$\varepsilon < \delta \leq \alpha$, so $F(\varepsilon + 1) \leq F(\delta) \leq \alpha$, or
$\delta + 1 \leq \varepsilon$, so $\alpha < F(\delta + 1) \leq F(\varepsilon)$.

Theorem: Fixed points

Definition

An argument x is a fixed-point for a function F if $F(x) = x$.

Theorem (Fixed-point theorem for normal functions)

Let F be a normal function. For every β there is an $\alpha > \beta$ such that $F(\alpha) = \alpha$.

Furthermore, the fixed-point α the theorem constructs is the least fixed-point greater than β.

Proof

☞ Define a function $f : \omega \to \mathbf{ON}$ by primitive recursion.

$$\begin{aligned} f(0) &= \beta + 1 \\ f(n+1) &= F(f(n)) \\ \alpha &= \sup\{f(n) \mid n \in \omega\}. \end{aligned}$$

☞ Suppose $f(0)$ is a fixed-point. Then $f(0) = F(f(0)) = f(1)$. By a simple induction, $f(0) = f(n)$ for all n. So, $\alpha = f(0)$ is a fixed-point.

☞ Suppose $f(0)$ is not a fixed point. Then $f(0) < F(f(0)) = f(1)$. By a simple induction, $f(n) < f(n+1)$ for all n (using F is order-preserving).
So, α is a limit ordinal, and a fixed-point:

$$\begin{aligned} F(\alpha) &= \sup\{F(\xi) \mid \xi < \alpha\} \\ &= \sup\{F(f(n)) \mid n \in \omega\} \\ &= \sup\{f(n+1) \mid n \in \omega\} \\ &= \alpha. \end{aligned}$$

Proof – continued

☞ Suppose $f(0) < \gamma < \alpha$. Then for some n,

$$f(n) \leq \gamma < f(n+1).$$

So,

$$f(n) \leq \gamma < f(n+1) \leq F(\gamma) < F(f(n+1)).$$

Thus, $\gamma \neq F(\gamma)$, so γ is not a fixed-point.

✓ α is the least fixed-point greater than β.

Example

Example. Define $\Lambda : \mathbf{ON} \to \mathbf{ON}$ be recursion:

$$\Lambda(\alpha) = \begin{cases} \omega & \text{if } \alpha = 0 \\ \Lambda(\beta) + \omega & \text{if } \alpha = S(\beta) \\ \sup\{\Lambda(\gamma) \mid \gamma < \alpha\} & \text{if } \alpha \text{ is a limit.} \end{cases}$$

Λ is enumerating the limit ordinals. To see this verify that

(i) $\alpha + \omega$ is smallest limit ordinal greater than α.

(ii) Λ is normal (slide 5), so that if γ is a limit ordinal, then so is $\Lambda(\gamma)$ (slide 9).

In Homework 9, you will be essentially proving that

$$\Lambda(\alpha) = \omega \cdot \alpha.$$

By the previous theorem, there is an $0 < \alpha$ (in fact, many α) with

$$\Lambda(\alpha) = \alpha = \omega \cdot \alpha.$$

Example

Example. The function $(\xi \mapsto \omega^\xi)$ is normal (noted previously, and proven in Lecture 21). So, there exists an $\alpha > 0$ (in fact, many α) with

$$\alpha = \omega^\alpha.$$

☞ In HW9 you will prove that for any $\beta < \alpha$

$$\beta + \alpha = \alpha.$$

Such numbers are called indecomposable, or γ-numbers .

☞ α has a stronger closure property:

$$\alpha = \omega^\alpha = \omega^{\omega^\alpha},$$

which implies that for any $\beta < \alpha$

$$\beta \cdot \alpha = \alpha.$$

Such numbers are called δ-numbers. See next Section.

γ-numbers

☞ Transfinite ordinals have properties that you do not find in finite number. The following property describes ordinals which absorb sums of smaller ordinals (on the left).

Definition

An ordinal α is called a γ-number if $\beta + \alpha = \alpha$ for all $\beta < \alpha$.

γ-numbers are also called additively indecomposable.

This is the terminology in Homework 9.

Note. $\alpha < \alpha + \beta$ when $\beta > 0$, so the order of the sum matters.

You can easily verify that 0 is a γ-number, and that ω is the next largest γ-number.

γ-numbers

☞ The following theorem is a problem on Homework 9.

Theorem

The following are equivalent.

1. α *is a* γ*-number:* $\beta + \alpha = \alpha$ *for all* $\beta < \alpha$*.*
2. *For all* $\beta, \gamma < \alpha$*,* $\beta + \gamma < \alpha$*.*
3. *Either* $\alpha = 0$*, or* $\alpha = \omega^\beta$ *for some* β

δ-numbers

☞ Transfinite ordinals have properties that you do not find in finite number. The following property describes ordinals which absorb products of smaller ordinals (on the left).

Definition

An ordinal α is called a δ-number if $\beta \cdot \alpha = \alpha$ for all $0 < \beta < \alpha$.
δ-numbers are also called multiplicatively indecomposable.

Note. $\alpha < \alpha \cdot \beta$ when $\beta > 1$ and $\alpha > 0$, so the order of the product matters.

You can easily verify that $0, 1, 2$ are δ-numbers, and ω is the next larger γ-number.

δ-numbers

☞ The proof of the following theorem is similar to the previous theorem on γ-numbers.

Theorem

The following are equivalent.

1. α *is a* δ*-number:* $\beta \cdot \alpha = \alpha$ *for all* $0 < \beta < \alpha$.
2. *For all* $\beta, \gamma < \alpha$, $\beta \cdot \gamma < \alpha$.
3. *Either* $\alpha = 0, 1, 2$, *or* $\alpha = \omega^{\omega^{\beta}}$ *for some* β

ϵ-numbers

☞ Transfinite ordinals have properties that you do not find in finite number. The following property describes ordinals which absorb exponents of smaller ordinals.

Definition

An ordinal α is called an ϵ-number if $\alpha^\beta = \alpha$ for all $1 < \beta < \alpha$.

Note. $\alpha \leq \beta^\alpha$ when $\beta > 1$, so the order of the exponent matters.

You can easily verify that $0, 1, 2$ are ϵ-numbers, and ω is the next larger ϵ-number.

Fixed points of exponentiation

☞ Since $(\xi \mapsto \omega^\xi)$ is normal, it has fixed points, $\alpha = \omega^\alpha$. These fixed points will be δ-numbers and γ numbers, from the characterization of slides 23 and 25:

$$\alpha = \omega^\alpha = \omega^{\omega^\alpha}.$$

☞ This fixed point is also an ϵ-number. Let $\beta < \alpha = \omega^\alpha$. Since α is a limit, there is a $\gamma < \alpha$ with $\beta < \omega^\gamma$. Then

β^α	$\leq$	$(\omega^\gamma)^\alpha$	L. 21, s. 17, (d)
	$=$	$\omega^{\gamma\cdot\alpha}$	L. 21, s. 18, (b)
	$=$	$\omega^\alpha = \alpha$	α is δ-number.

☞ We turn to finding a nice characterization of ϵ numbers.

Knuth Double Arrow notation

☞ Extend Knuth's "double arrow" operator to the transfinite:

$$\beta \uparrow\uparrow \alpha = \underbrace{\beta^{\beta^{\beta^{\cdot^{\cdot^{\cdot}}}}}}_{\alpha \text{ copies}}$$

Definition (Ordinal Double Arrow)

For all ordinals β,

$$\begin{aligned} \beta \uparrow\uparrow 0 &= 1 \\ \beta \uparrow\uparrow (\alpha + 1) &= (\beta \uparrow\uparrow \alpha)^{\beta} \\ \beta \uparrow\uparrow \alpha &= \sup\{\beta \uparrow\uparrow \xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

Special Topics

Fixed points and ϵ_0

Example. The least fixed point of the normal function $(\xi \mapsto \omega^{\xi})$ is

$$\epsilon_0 = \omega \uparrow\uparrow \omega = \underbrace{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}_{\omega \text{ copies}}$$

You can check this by looking back to the construction in the proof of the Fixed-Point Theorem on slide 16.

☞ The function $(\xi \mapsto \omega \uparrow\uparrow \xi)$ is normal, so there are (many) fixed points

$$\alpha = \omega \uparrow\uparrow \alpha = \underbrace{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}}_{\alpha \text{ copies}}$$

and these will be ϵ-numbers.

☞ Not all ϵ-numbers are fixed points of $\uparrow\uparrow$, for example $\epsilon_0 \neq \omega \uparrow\uparrow \epsilon_0$. (In fact, $\epsilon_0 \neq \epsilon_0^{\omega} = \omega \uparrow\uparrow (\omega + 1)$ – see L. 21, s. 17, (c).)

ϵ-numbers

Theorem

The following conditions are equivalent

(a) α *is an* ϵ*-number:* $\beta^\alpha = \alpha$ *for all* $\beta < \alpha$*.*

(b) $\alpha = 1$*, or for all* $\beta, \gamma < \alpha$*,* $\beta^\gamma < \alpha$*.*

(c) $\alpha = \beta \uparrow\uparrow \omega$ *for some* β*.*

Note. For (c),

$$\begin{aligned} 1 &= 1 \uparrow\uparrow \beta && \text{for all } \beta \\ \omega &= n \uparrow\uparrow \omega && \text{for all } n \in \omega. \end{aligned}$$