Math 582
Introduction to Set Theory

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January 9, 2009

Set Union

Definition. Let A and B be sets. The union of A and B is the set

$$A \cup B = \{x \mid x \in A \ \vee \ x \in B\}$$

☞ By Comprehension this set exists for each A and B.

Examples.

$$\begin{aligned} \{\clubsuit, \spadesuit\} \cup \{\heartsuit, \diamondsuit\} &= \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\} \\ \{0, 2, 4, \ldots\} \cup \{1, 3, 5, \ldots\} &= \mathbb{N}. \end{aligned}$$

Basic properties of union

Proposition. Let A, B and C be sets. The following properties hold:

- Idempotency $A \cup A = A$,
- Commutativity $A \cup B = B \cup A$,
- Associativity $(A \cup B) \cup C = A \cup (B \cup C)$.

Union and subset

Proposition. Let A and B be sets. Then the following are equivalent

$$(a) \quad A \subseteq B,$$
$$(b) \quad A \cup B = B.$$

Set Intersection

Definition. Let A and B be sets. The intersection of A and B is the set

$$A \cap B = \{x \,|\, x \in A \wedge x \in B\}$$

☞ By Comprehension this set exists for each A and B.

Examples.

$$\begin{aligned} \{\clubsuit, \spadesuit\} \cap \{\heartsuit, \diamondsuit\} &= \emptyset \\ \{0, 2, 4, \ldots\} \cap \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} &= \{0, 2, 4, 6, 8\} \\ \mathbb{N} \cap \mathbb{Q} &= \mathbb{N}. \end{aligned}$$

Basic properties of intersection

Proposition. Let A, B and C be sets. The following properties hold:

Idempotency $A \cap A = A$,

Commutativity $A \cap B = B \cap A$,

Associativity $(A \cap B) \cap C = A \cap (B \cap C)$.

Intersection and subset

Proposition. Let A and B be sets. Then the following are equivalent

$$\begin{array}{ll} (a) & A \subseteq B \\ (b) & A \cap B = A. \end{array}$$

Intersection and Union

Proposition. Let A, B and C be sets. The following properties hold:

- Absorption of $\cup$ over $\cap$ $\quad A \cup (A \cap B) = A$,
- Absorption of $\cap$ over $\cup$ $\quad A \cap (A \cup B) = A$,
- Distributivity of $\cup$ over $\cap$ $\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
- Distributivity of $\cap$ over $\cup$ $\quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Set Difference

Definition. Let A and B be sets. The difference of A and B is the set

$$A - B = \{x \mid x \in A \ \wedge \ x \notin B\}$$

where $x \notin B$ abbreviates $\neg(x \in B)$.

☞ By Comprehension this set exists for each A and B.

Examples.

$$\begin{aligned} \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\} - \{\clubsuit, \spadesuit\} &= \{\heartsuit, \diamondsuit\} \\ \mathbb{N} - \mathbb{Z} &= \emptyset \\ \mathbb{R} - \mathbb{Q} &= \{x \mid x \text{ is irrational }\}. \end{aligned}$$

Difference and subset

Proposition. Let A and B be sets. Then the following are equivalent

$$\begin{array}{ll} (a) & A \subseteq B \\ (b) & A - B = \emptyset. \end{array}$$

De Morgan's laws

Proposition. Let A, B and C be sets. The following properties hold:

De Morgan's Law for $\cup$ $\quad A - (B \cup C) = (A - B) \cap (A - C)$,

De Morgan's Law for $\cap$ $\quad A - (B \cap C) = (A - B) \cup (A - C)$,

Sets as objects

☞ Up to now we have used the Comprehension principle to gather objects to form a set. However, this is no reason why these *objects* could not be themselves sets.

Examples. The following sets exist by the Comprehension Principle:

$$\emptyset, \quad \{\emptyset\}, \quad \{\emptyset, \{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

For example, since $\emptyset$ is a uniquely defined object,

$$\begin{aligned} \{\emptyset\} &= \{x \mid x = \emptyset\} \\ \{\emptyset, \{\emptyset\}\} &= \{x \mid x = \emptyset \ \lor \ x = \{\emptyset\}\} \\ \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} &= \{x \mid x = \emptyset \ \lor \ x = \{\emptyset\} \ \lor \ \{\emptyset, \{\emptyset\}\}\}. \end{aligned}$$

Operations on sets

☞ There are also operations which can produce sets of sets.

Examples.

- The singleton operator is defined by $\text{SING}(x) = \{x\}$. For example,

$$\text{SING}(\clubsuit) = \{\clubsuit\} \quad \text{SING}(\emptyset) = \{\emptyset\}$$

- The pairing operator is defined by $\text{PAIR}(x, y) = \{x, y\}$. For example,

$$\text{PAIR}(\diamondsuit, \emptyset) = \{\diamondsuit, \emptyset\} \quad \text{PAIR}(\mathbb{N}, \mathbb{N}) = \{\mathbb{N}\}.$$

- We can combine these to produce new operations:

$$\text{OPAIR}(x, y) = \text{PAIR}(\text{SING}(x), \text{PAIR}(x, y)) = \{\{x\}, \{x, y\}\}.$$

A curiosity

Question. Is it possible for $x = \{x\}$? For example,

$$\clubsuit \neq \{\clubsuit\} \quad \emptyset \neq \{\emptyset\} \quad \mathbb{R} \neq \{\mathbb{R}\} \quad \{\diamondsuit, \pi\} \neq \{\{\diamondsuit, \pi\}\}.$$

☞ What about

$$\{\{\{\ldots\emptyset\ldots\}\}\}.$$

which has infinitely many nested brackets? Is it a set?

Operations on sets

Definition. If A is any set, the collection of *all* subsets of A is called the power set of A, written as

$$\mathcal{P}(A) = \{B \mid B \subseteq A\}.$$

Examples.

$$\begin{aligned} \mathcal{P}(\emptyset) &= \{\emptyset\} \\ \mathcal{P}(\{\clubsuit, \diamondsuit\}) &= \{\emptyset, \{\clubsuit\}, \{\diamondsuit\}, \{\clubsuit, \diamondsuit\}\}. \end{aligned}$$

Remark. $\emptyset, A \in \mathcal{P}(A)$ for any set A.

A curiosity

Question. Is it possible for a set A that $A = \mathcal{P}(A)$? For example,

$$\{\clubsuit\} \neq \{\emptyset, \{\clubsuit\}\} = \mathcal{P}(\{\clubsuit\}) \qquad \{0, 1\} \neq \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} = \mathcal{P}(\{0, 1\}) \qquad \mathbb{N} \neq \mathcal{P}(\mathbb{N}).$$

For every finite set A: $A \neq \mathcal{P}(A)$.

Argue by size. If A has n elements, then $\mathcal{P}(A)$ has 2^n elements.

Is this also true for all infinite sets?

☞ Recall, $V = \{x \mid x = x\}$.
Does $V = \mathcal{P}(V)$?

Generalized Union

Convention. We will say a set A is a family of sets if it is a set, all of whose members are sets.

Definition. Let A be a family of sets. The union of A is the set

$$\bigcup A = \{a \mid \exists x (x \in A \ \wedge \ a \in x)\},$$

the set consisting of all elements of sets of A.

We extend our logical notation and write

$\exists x \in A$ means 'there exists an x in A such that',

so, that we may re-write

$$\bigcup A = \{a \mid \exists x \in A (a \in x)\}.$$

Generalized Intersection

Definition. Let A be a family of sets. The intersection of A is the set

$$\bigcap A = \{a \mid \forall x (x \in A \ \rightarrow \ a \in x)\},$$

the set consisting of all elements in every set in A.

We extend our logical notation and write

$\forall x \in A$ means 'for every x in A' ,

so, that we may re-write

$$\bigcap A = \{a \mid \forall x \in A (a \in x)\}.$$

Simple facts

Proposition. Let A and B be sets. The following hold.

$$A \cup B = \bigcup \{A, B\} \qquad A \cap B = \bigcap \{A, B\},$$

and

$$\bigcup \{A\} = A \qquad \bigcap \{A\} = A.$$