Math 582
Introduction to Set Theory
Lecture 19

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March 9, 2009

Supremum

☞ HW4 (Week 8, Exercise 4): Let $X \subset \textbf{ON}$ be a set.
Then $\bigcup X$ is an ordinal which is greater than or equal to all elements of X:

(a) $\bigcup X \geq \alpha$ for all $\alpha \in X$ (an upper bound),

(b) If $\gamma \geq \alpha$ for all $\alpha \in X$ then $\bigcup X \leq \gamma$ (least among upper bounds.)

☞ We denote $\bigcup X$ by $\sup X$, the supremum of X.

Limit and Successor Ordinals

☞ The following trivial lemma summarizes the important properties of successor and limit ordinals.

Lemma

*Let $\alpha \in$ **ON**.*

❶ *α is a limit ordinal if and only if $\alpha \neq 0$ and $\{\beta \mid \beta < \alpha\}$ has no largest member.*
In this case, $\alpha = \sup \alpha = \sup\{\beta \mid \beta < \alpha\}$.

❷ *α is a successor ordinal if and only if $\{\beta \mid \beta < \alpha\}$ has a largest member.*
In this case, $\alpha = S(\beta)$, where β is largest in $\{\beta \mid \beta < \alpha\}$ and $\sup \alpha = \beta$.

Supremum

Lemma

*Let $X, Y \subseteq$ **ON**.*
If for every $\alpha \in X$ there is a $\beta \in Y$ with $\alpha \leq \beta$, then $\sup X \leq \sup Y$.

Additionally,
if for every $\beta \in Y$ there is an $\alpha \in X$, then $\sup X = \sup Y$.

Proof.

☞ Suppose for each $\alpha \in X$, there is a $\beta \in Y$ with $\alpha \leq \beta$.
Since $\sup Y$ is an upper bound of Y, $\alpha \leq \sup Y$ for each $\alpha \in X$ ($\sup Y$ is an upper bound of Y).

Therefore, $\sup X \leq \sup Y$, since $\sup X$ is the least upper bound of X.

☞ The second part follows by the antisymmetry of $\leq$ on **ON**, since then $\sup X \leq \sup Y$ and $\sup Y \leq \sup X$. □

Addition on **ON**

Definition (Ordinal addition)

For all ordinals β,

$$\begin{aligned} \beta + 0 &= \beta \\ \beta + S(\alpha) &= S(\beta + \alpha) \\ \beta + \alpha &= \sup\{\beta + \xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

Formal justification

Formally, for each β, define $\mathbf{G}_\beta : \mathbf{V} \to \mathbf{V}$ so that

(i) $\mathbf{G}_\beta(t) = 0$ unless t is a function and $\text{dom}(t) = \alpha \in \mathbf{ON}$,

(ii) $\mathbf{G}_\beta(t) = \beta$ when $\alpha = 0$,

(iii) $\mathbf{G}_\beta(t) = S(t(\delta))$ when $\alpha = S(\delta)$,

(iv) $\mathbf{G}_\beta(t) = \sup\{t(\xi) \mid \xi < \alpha\}$ when α is a limit.

$\mathbf{G}_\beta$ is just a formula $\varphi(\beta, x, y)$ where β is a parameter, x is the argument to $\mathbf{G}_\beta$ and y is the value. Since each of the above four cases is mutually distinct and collectively exhaustive, for each β we can prove $\forall x \exists ! y \varphi(\beta, x, y)$. So, $\mathbf{G}_\beta$ is a class function for each β.

The Transfinite Recursion Theorem gives a unique class function $+_\beta : \mathbf{ON} \to \mathbf{V}$ for each β satisfying the clauses of ordinal addition: $+_\beta(\alpha) = \beta + \alpha$.

Consequences

$$\begin{aligned} \beta + 0 &= \beta \\ \beta + S(\alpha) &= S(\beta + \alpha) \\ \beta + \alpha &= \sup\{\beta + \xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

- ♣ Let $\alpha = 1$ and we get $\beta + 1 = S(\beta)$. (I'll write $\beta + 1$ for $S(\beta)$.)
- ♣ Ordinal addition agrees with addition on natural numbers (only the 0 and successor clauses are relevant.)
- ♣ $\omega < \omega + 1 < \omega + 2 < \omega + 3 < \ldots < \omega + n < \ldots$ (for $n < \omega$.)
- ♣ $\omega + \omega = \sup\{\omega + n \mid n \in \omega\}$, and is a limit ordinal.
- ♣ $1 + \omega = \sup\{1 + n \mid n < \omega\} = \omega$, $n + \omega = \omega$ for every $n < \omega$.
- ♣ So, $1 + \omega \neq \omega + 1$ (ordinal addition is not commutative.)

Successor and addition

Lemma

For every ordinal α,

$$S(\alpha) = \alpha + 1.$$

From now on I will write '$\alpha + 1$' for '$S(\alpha)$'.

Proof.

$$\alpha + 1 = S(\alpha + 0) = S(\alpha).$$

□

Order Laws for addition

Lemma (Order)

Suppose $\beta < \gamma$. Then for every α

(a) $\alpha + \beta < \alpha + \gamma$,

(b) $\beta + \alpha \leq \gamma + \alpha$ *and* $\leq$ *cannot be replaced by* $<$.

☞ To see the last clause in (b):
$1 < 2$ but $1 + \omega = \omega = 2 + \omega$.

Proof of (a)

(a). Proof by transfinite induction on γ.
We will show that $\beta < \gamma$ implies $\alpha + \beta < \alpha + \gamma$.

Suppose $\beta < \gamma$ and that $\alpha + \beta < \alpha + \delta$ whenever $\beta < \delta < \gamma$ (the inductive hypothesis.)

The proof breaks into the successor case and the limit case ($\gamma = 0$ cannot arise.)

☞ $\gamma = \delta + 1$. Then, $\beta \leq \delta$ and $\alpha + \beta \leq \alpha + \delta$ (by i.h.), so

$$\begin{array}{rcll} \alpha + \beta & \leq & \alpha + \delta & \text{i.h.} \\ & < & (\alpha + \delta) + 1 & \\ & = & \alpha + (\delta + 1) & \text{def. of } + \\ & = & \alpha + \gamma. & \end{array}$$

Proof of (a) continued

Inductive Hypothesis: $\alpha + \beta < \alpha + \delta$ whenever $\beta < \delta < \gamma$.

☞ γ is a limit. Since $\beta < \gamma$ and γ is a limit ordinal (there is no greatest ordinal in γ), there is a $\delta < \gamma$ with $\beta < \delta < \gamma$. So,

$$\begin{aligned} \alpha + \beta &< \alpha + \delta && \text{i.h.} \\ &\leq \sup\{\alpha + \xi \,|\, \xi < \gamma\} && \text{sup an upper-bound} \\ &= \alpha + \gamma && \text{def. of } + \end{aligned}$$

✓ So, $\alpha + \beta < \alpha + \gamma$ whenever $\beta < \gamma$ by transfinite induction.

Proof of (b)

(b). Suppose $\beta < \gamma$. Proof by transfinite induction on α.
We will show that $\beta + \alpha \leq \gamma + \alpha$.

☞ $\alpha = 0$. $\beta + 0 = \beta < \gamma = \gamma + 0$.

☞ $\alpha = S(\delta)$. Then, $\beta + \delta \leq \gamma + \delta$ (i.h.) implies

$$\beta + (\delta + 1) = (\beta + \delta) + 1 \leq (\gamma + \delta) + 1 = \gamma + (\delta + 1).$$

☞ α is a limit. Then, $\beta + \xi \leq \gamma + \xi$ for all $\xi < \alpha$. So

$$\sup\{\beta + \xi \,|\, \xi \leq \alpha\} \leq \sup\{\gamma + \xi \,|\, \xi < \alpha\};$$

that is, $\beta + \alpha \leq \gamma + \alpha$ by definition of addition.

Left Cancellation law

Lemma (Left Cancellation)

$\alpha + \beta = \alpha + \gamma$ *implies* $\beta = \gamma$, *for all* α, β, γ.

☞ The Right Cancellation law is **not** generally true:
$\beta + \alpha = \gamma + \alpha$ does not necessarily imply $\beta = \gamma$:
$1 + \omega = 2 + \omega$ but $1 \neq 2$.

Proof of Lemma

Suppose $\alpha + \beta = \alpha + \gamma$.
Then exactly one of $\beta < \gamma$ or $\gamma < \beta$ or $\beta = \gamma$.

☞ If $\beta < \gamma$ then $\alpha + \beta < \alpha + \gamma$ by the Order Lemma (a).

☞ If $\gamma < \beta$ then $\alpha + \gamma < \alpha + \beta$ by the Order Lemma (a).

✓ Therefore, it must be that $\beta = \gamma$.

Associativity of Addition

Lemma (Associativity)

$(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ *for all* α,β,γ.

Proof of Associativity

The proof is by transfinite induction on γ.

☞ $\gamma=0$. Then

$$(\alpha+\beta)+0=\alpha+\beta=\alpha+(\beta+0)$$

☞ $\gamma=\delta+1$. Assume (i.h.) that $(\alpha+\beta)+\delta=\alpha+(\beta+\delta)$. Then

$$\begin{aligned}(\alpha+\beta)+\gamma &= (\alpha+\beta)+(\delta+1)\\ &= \big((\alpha+\beta)+\delta\big)+1\\ &= \big(\alpha+(\beta+\delta)\big)+1 && \text{i.h.}\\ &= \alpha+\big((\beta+\delta)+1\big)\\ &= \alpha+(\beta+(\delta+1))\\ &= \alpha+(\beta+\gamma)\end{aligned}$$

Proof of Associativity continued

☞ Suppose γ is a limit. Assume (i.h.) that $(\alpha+\beta)+\delta = \alpha+(\beta+\delta)$ for all $\delta < \gamma$.

First, $\beta+\gamma = \text{sup}\{\beta+\delta \mid \delta < \gamma\}$, and this is limit ordinal by the Order lemma – if $\delta < \delta'$ then $\beta+\delta < \beta+\delta'$, so that $\{\beta+\delta \mid \delta<\gamma\}$ has no greatest element.

Then,

$$\begin{aligned}(\alpha+\beta)+\gamma &= \text{sup}\{(\alpha+\beta)+\delta \mid \delta<\gamma\} \\ &= \text{sup}\{\alpha+(\beta+\delta) \mid \delta<\gamma\} && \text{i.h.} \\ &= \text{sup}\{\alpha+\xi \mid \xi < \beta+\gamma\} && ❊ \\ &= \alpha+(\beta+\gamma) && \beta+\gamma \text{ a limit}\end{aligned}$$

We finish the case with a proof of ❊.

Proof of Associativity continued

Proof of ❊ $\text{sup}\{\alpha+(\beta+\delta) \mid \delta<\gamma\} = \text{sup}\{\alpha+\xi \mid \xi<\beta+\gamma\}$

(i) If $\delta<\gamma$, then $\beta+\delta<\beta+\gamma$ by the Order lemma,
so $\alpha+(\beta+\delta) \in \{\alpha+\xi \mid \xi<\beta+\gamma\}$;

(ii) If $\xi<\beta+\gamma$ then $\xi<\beta+\delta$ for some $\delta<\gamma$ (since $\beta+\gamma$ is a limit.)
By the Order lemma

$$\alpha+\xi < \alpha+(\beta+\delta) \in \{\alpha+(\beta+\delta) \mid \delta<\gamma\}.$$

Thus, from (i) and (ii) it follows that ❊

$$\text{sup}\{\alpha+(\beta+\delta) \mid \delta<\gamma\} = \text{sup}\{\alpha+\xi \mid \xi<\beta+\gamma\}.$$

✓ Therefore, $(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ for all α,β,γ.

Subtraction

Lemma (Subtraction)

If $\alpha \leq \beta$ then there is a unique ordinal γ such that $\alpha + \gamma = \beta$.

Proof of Subtraction Lemma

Fix α. The proof is by transfinite induction on β. The theorem trivially holds when $\beta < \alpha$. By Left Cancellation it is sufficient to prove the existence of ξ.

- ☞ $\beta = \alpha$. Then $\alpha + 0 = \beta$.
- ☞ $\beta = \delta + 1$ and $\alpha < \beta$. Assume (i.h.) that $\alpha + \xi = \delta$. Then
  $\alpha + (\xi + 1) = (\alpha + \xi) + 1 = \delta + 1 = \beta$.
- ☞ β a limit and $\alpha < \beta$. Let $\eta = \sup\{\xi \mid \exists \gamma < \beta\, (\alpha + \xi = \gamma)\}$. Then

$$\begin{aligned} \beta &= \sup\{\gamma \mid \alpha \leq \gamma < \beta\} && \beta \text{ is a limit} \\ &= \sup\{\alpha + \xi \mid \alpha + \xi < \beta\} && \text{i.h.} \\ &= \sup\{\alpha + \xi \mid \xi < \eta\} && ❷ \\ &= \alpha + \eta && \eta \text{ is a limit, } ❶ \end{aligned}$$

✓ The proof is completed once we bridge the gap with ❶ and ❷ .

Two points in proof

❶ $\eta = \sup\{\xi \mid \exists\gamma < \beta\,(\alpha + \xi = \gamma)\}$ is a limit.

Let $\xi < \eta$, so that there is a δ with $\xi \leq \delta \leq \eta$ and $\alpha + \delta < \beta$. Since β is a limit, by the i.h. there is a $\delta'' > \delta' > \delta$ with

$$\alpha + \delta < \alpha + \delta' < \alpha + \delta'' < \beta.$$

By the Order Lemma (a), $\xi \leq \delta < \delta' < \eta$. So, η has no greatest element.

❷ $\sup\{\alpha + \xi \mid \alpha + \xi < \beta\} = \sup\{\alpha + \xi \mid \xi < \eta\}$.

If $\alpha + \xi < \beta$ then $\xi < \eta$. On the other hand, if $\xi < \eta$, then there is some $\delta \geq \xi$ with $\delta \in \eta$, so

$$\alpha + \xi < \alpha + \delta < \beta.$$

Multiplication on **ON**

Definition (Ordinal multiplication)

For all ordinals β,

$$\begin{aligned} \beta \cdot 0 &= 0 \\ \beta \cdot (\alpha + 1) &= \beta \cdot \alpha + \beta \\ \beta \cdot \alpha &= \sup\{\beta \cdot \xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

Formal justification

Formally, for each β, define $\mathbf{G}_\beta : \mathbf{V} \to \mathbf{V}$ (a formula in the language of set theory) so that

(i) $\mathbf{G}_\beta(t) = 0$ unless t is a function and $\text{dom}(t) = \alpha \in \mathbf{ON}$,

(ii) $\mathbf{G}_\beta(t) = 0$ when $\alpha = 0$,

(iii) $\mathbf{G}_\beta(t) = t(\delta) + \beta$ when $\alpha = S(\delta)$,

(iv) $\mathbf{G}_\beta(t) = \sup\{t(\xi) \mid \xi < \alpha\}$ when α is a limit.

$\mathbf{G}_\beta$ is a formula $\varphi(\beta, x, y)$ where β is a parameter, x is the argument to $\mathbf{G}_\beta$ and y is the value. Since each of the above four cases is mutually distinct and collectively exhaustive, for each β we can prove $\forall x \exists ! y \varphi(\beta, x, y)$. So, $\mathbf{G}_\beta$ is a class function for each β.

The Transfinite Recursion Theorem gives a unique class function $\cdot_\beta : \mathbf{ON} \to \mathbf{V}$ for each β satisfying the clauses of ordinal addition: $\cdot_\beta(\alpha) = \beta \cdot \alpha$.

Examples of Multiplication

$$\begin{aligned} \beta \cdot 0 &= 0 \\ \beta \cdot (\alpha + 1) &= \beta \cdot \alpha + \beta \\ \beta \cdot \alpha &= \sup\{\beta \cdot \xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

- ♣ $\beta \cdot 1 = \beta \cdot (0 + 1) = 0 + \beta = \beta$,
- ♣ $\beta \cdot 2 = \beta \cdot 1 + \beta = \beta + \beta$,
- ♣ $\beta \cdot 3 = \beta \cdot 2 + \beta = \beta + \beta + \beta$,
- ♣ $\beta < \beta \cdot 2 < \beta \cdot 3 < \beta \cdot 4$
- ♣ $\beta \cdot \omega = \sup\{\beta \cdot n \mid n < \omega\}$ is a limit ordinal.
- ♣ $1 \cdot \omega = \sup\{1 \cdot n \mid n < \omega\} = \omega$,
- ♣ $2 \cdot \omega = \sup\{2 \cdot n \mid n < \omega\} = \omega$,
- ♣ So, $2 \cdot \omega \neq \omega \cdot 2$. Thus, multiplication is not commutative.

Exponentiation on **ON**

Definition (Ordinal exponentiation)

For all ordinals β,

$$\begin{aligned} \beta^0 &= 1 \\ \beta^{\alpha+1} &= \beta^\alpha \cdot \beta \\ \beta^\alpha &= \sup\{\beta^\xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit} \end{aligned}$$

Formal justification

Formally, for each β, define $\mathbf{G}_\beta : \mathbf{V} \to \mathbf{V}$ (a formula in the language of set theory) so that

(i) $\mathbf{G}_\beta(t) = 0$ unless t is a function and $\text{dom}(t) = \alpha \in \mathbf{ON}$,

(ii) $\mathbf{G}_\beta(t) = 1$ when $\alpha = 0$,

(iii) $\mathbf{G}_\beta(t) = t(\delta) \cdot \beta$ when $\alpha = S(\delta)$,

(iv) $\mathbf{G}_\beta(t) = \sup\{t(\xi) \mid \xi < \alpha\}$ when α is a limit.

$\mathbf{G}_\beta$ is a formula $\varphi(\beta, x, y)$ where β is a parameter, x is the argument to $\mathbf{G}_\beta$ and y is the value. Since each of the above four cases is mutually distinct and collectively exhaustive, for each β we can prove $\forall x \exists! y \varphi(\beta, x, y)$. So, $\mathbf{G}_\beta$ is a class function for each β.

The Transfinite Recursion Theorem gives a unique class function $\mathbf{F}_\beta : \mathbf{ON} \to \mathbf{V}$ for each β satisfying the clauses of ordinal addition: $\mathbf{F}_\beta(\alpha) = \beta^\alpha$.

Examples of Multiplication

$$\beta^0 = 1$$
$$\beta^{\alpha+1} = \beta^\alpha \cdot \beta$$
$$\beta^\alpha = \sup\{\beta^\xi \mid \xi < \alpha\} \qquad \text{when } \alpha \text{ is a limit}$$

- ♣ $\beta^1 = \beta$, $\beta^2 = \beta \cdot \beta$, $\beta^3 = \beta \cdot \beta \cdot \beta$,
- ♣ $\beta^\omega = \sup\{\beta^n \mid n < \omega\}$
- ♣ $1^\beta = 1$ for all β,
- ♣ $2^\omega = \sup\{2^n \mid n < \omega\} = \omega$,
- ♣ $3^\omega = \omega$
- ♣ $\sup\{n^\omega \mid n < \omega\} = \omega \neq \omega^\omega = \sup\{\omega^n \mid n < \omega\}$

Initial segment of the ordinal numbers

We can continue counting into the transfinite:

$0, 1, 2, \ldots, \omega, \omega{+}1, \omega{+}2, \ldots, \omega{\cdot}2, \ldots, \omega{\cdot}3, \ldots, \omega{\cdot}\omega, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \omega_1, \ldots$

Eventually we reach the first ordinal number after ω which has a greater cardinality than ω; this uncountable ordinal is ω_1 and its cardinality is $\aleph_1$.

All ordinals we have generated so far, ω^ω, ω^{ω^ω} etc., are countable ordinals. The existence of ω_1 depends on a new axiom, the Power Set Axiom, which we will introduce shortly.