Math 582
Introduction to Set Theory

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Introduction

We will extend the two fundamental principles on the natural numbers to the whole class of ordinals: Induction, a principle of proof, and Recursion, a principle of definition. There are two obstacles in the formulation of these principles to overcome:

✈ There are limit ordinals as well as successor ordinals; there are only successors with the natural numbers.

✈ **ON** is a proper class, so there is no hope for a set formulation of these principles, as we had with the natural numbers.

Once we overcome these obstacles we will define addition, multiplication and exponentiation on the ordinals, and use this added structure to improve our understanding of the structure of **ON**.

Transfinite Induction I

The primary difficulty with Transfinite Induction is the presence of limit ordinals. For this reason we generalize the *Complete Induction Principle* on natural numbers.

Theorem (Transfinite Induction Principle on ON)

Let $\varphi(x)$ be a formula. Suppose the following holds

$$\forall\alpha\,(\forall\xi<\alpha\,\varphi(\xi)\ \rightarrow\ \varphi(\alpha))\qquad\ast$$

Then $\forall\alpha\,\varphi(\alpha)$.

Note. Formally, this is a theorem scheme – a different theorem for each formula φ.

Proof of Transfinite Induction Principle

Proof.

We are assuming

$$\forall\alpha\,(\forall\xi<\alpha\,\varphi(\xi)\ \rightarrow\ \varphi(\alpha))\qquad\ast$$

☞ Suppose there is an ordinal α for which $\neg\varphi(\alpha)$.
Fix such a α and let X be the set

$$X=\{\gamma\leq\alpha\,|\,\neg\varphi(\gamma)\}$$

Since **ON** is well-ordered, there is a least $\gamma\in X$ with $\neg\varphi(\gamma)$.
But then $\forall\xi<\gamma\,\varphi(\xi)$ is true, so by $\ast$ we have $\varphi(\gamma)$. ↯

✓ $\forall\alpha\,\varphi(\alpha)$. □

Transfinite Induction Principle II

☞ It is often more convenient to separate limits and successors as separate cases, when applying Transfinite Induction.

The following is just like Induction on ω with limit ordinals as a third case.

Theorem (Transfinite Induction Principle on ON)

Let $\varphi(x)$ be a formula. Suppose the following

(a) $\varphi(0)$ *holds.*

(b) $\forall\alpha\,(\varphi(\alpha) \rightarrow \varphi(S(\alpha)))$ *holds.*

(c) $\forall$ *limit* $\alpha\,(\forall\xi < \alpha\,\varphi(\xi) \rightarrow \varphi(\alpha))$ *holds.*

Then $\forall\alpha\,\varphi(\alpha)$.

Proof of Transfinite Induction Principle II

Proof.

We need to prove

$$\forall\alpha\,(\forall\xi < \alpha\,\varphi(\xi) \rightarrow \varphi(\alpha)) \qquad \ast$$

Three cases:

(i) $\alpha = 0$. The $\phi(0)$ is true by (a), so $\ast$ holds.

(ii) $\alpha = S(\beta)$. Suppose $\forall\xi < S(\beta)\,\varphi(\xi)$. In particular, since $\beta < S(\beta)$ we have $\varphi(\beta)$; so, $\varphi(S(\beta))$ by (b). Thus, $\ast$ holds.

(iii) α a limit ordinal. Then, $\ast$ holds by (c).

This establishes $\ast$, so we can apply Transfinite Induction I to conclude $\forall\alpha\,\varphi(\alpha)$. □

A word on functions

There are two notions of function at play in Transfinite Recursion:

① Set functions: these are functions as sets-of-ordered-pairs and are objects of set theory. We write $f : X \to Y$ to mean f is a set-of-ordered-pairs satisfying $(x, y), (x, z) \in f \ \to y = z$ with set domain X and set range Y.

② Class functions: these are functions as rules-associating-argument-to-value, and are **not actual sets**, but are statements in the language of set theory. We write $\mathbf{G} : \mathbf{V} \to \mathbf{V}$ to mean that there is a formula $\varphi(x, y)$ satisfying the following two conditions:

$$\forall x \exists ! y \, \varphi(x, y)$$
$$\mathbf{G}(x) = y \quad \leftrightarrow \quad \varphi(x, y)$$

You can think of **G** as a defined symbol in the language which abbreviates a statement in set theory; **G does not** denote an object (i.e. a set).

Transfinite Recursion on ON

Theorem

If $\mathbf{G} : \mathbf{V} \to \mathbf{V}$ then there is a unique $\mathbf{F} : \mathbf{ON} \to \mathbf{V}$ such that

$$\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F} \upharpoonright \alpha)$$

Note. $\mathbf{G}(x) = y$ is really an abbreviation of some formula $\varphi(x, y)$; and the formula $\varphi(x, y)$ may have other free variables, parameters, which play no role in the proof. These parameters are useful for defining functions, but are otherwise inert.

Notation for Transfinite Recursion

The statement of the Transfinite Recursion Theorem uses three pieces of notation:

① $\mathbf{G} : \mathbf{V} \to \mathbf{V}$. This is an abbreviation of the expression $\forall x \exists! y\, \varphi(x, y)$ for some formula $\varphi(x, y)$. The expression $\mathbf{G}(x) = y$ abbreviates the expression $\varphi(x, y)$.

② The Theorem claims that there is a formula $\psi(x, y)$ for which we can prove

(a) $\forall x \in \mathbf{ON}\, \exists! y\, \psi(x, y)$, so ψ defines a class function $\mathbf{F}$ whose domain is the ordinals, $\mathbf{F} : \mathbf{ON} \to \mathbf{V}$, and $\mathbf{F}(x) = y$ abbreviates $\psi(x, y)$;

(b) $\forall \xi \in \mathrm{ON}\, \mathbf{F}(\xi) = \mathbf{G}(\mathbf{F} \restriction \xi)$.

③ For any particular ordinal α, $\mathbf{F} \restriction \alpha$ is a set (whose existence is guaranteed by Replacement and Comprehension):

$$\mathbf{F} \restriction \alpha = \{(\xi, y) \,|\, \xi < \alpha \wedge \psi(\xi, y)\}$$

δ-approximations

For each ordinal δ let $\mathrm{APP}(\delta, h)$, h is a δ-approximation to $\mathbf{F}$, say

(i) h is a function (set-of-ordered-pairs),

(ii) $\mathrm{dom}(h) = \delta$ and

(iii) $h(\xi) = \mathbf{G}(h \restriction \xi)$ for all $\xi < \delta$.

(Compare δ-*approximations* to *computations* from the proof of the Recursion theorem for natural numbers.)

The ♡ is the proof of existence and uniqueness of δ-approximations:

- Existence: $\forall \delta\, \exists h\, \mathrm{APP}(\delta, h)$.
- Uniqueness: $\delta < \delta' \wedge \mathrm{APP}(\delta, h) \wedge \mathrm{APP}(\delta', h') \longrightarrow h = h' \restriction \delta$

Defining F

With Uniqueness and Existence we define **F** so that it satisfies

$$\mathrm{APP}(\delta, h) \quad \longleftrightarrow \quad \mathbf{F} \restriction \delta = h$$

We actually define a formula $\psi(x, y)$ using $\mathrm{APP}(\delta, h)$ by

$$\psi(x, y) \quad \longleftrightarrow \quad \big(x \notin \mathbf{ON} \wedge y = 0\big) \vee \big(x \in \mathbf{ON} \wedge \exists\delta > x \, \exists h \, [\mathrm{APP}(\delta, h) \wedge h(x) = y]\big).$$

Uniqueness and Existence say that for any ordinal δ, the family of functions $\{h_\xi \mid \mathrm{APP}(\xi, h_\xi) \wedge \xi < \delta\}$ is a compatible family of functions. Thus, $\bigcup_{\xi<\delta} h_\xi$ is a function (HW3, week 5.)

Think of **F** as "defined by"

$$\mathbf{F} = \bigcup_{\delta \in \mathbf{ON}} h_\delta.$$

Uniqueness

We prove Uniqueness by Transfinite Induction on **ON**:

$$\delta < \delta' \wedge \mathrm{APP}(\delta, h) \wedge \mathrm{APP}(\delta', h') \quad \longrightarrow \quad h = h' \restriction \delta$$

Suppose there are $\delta < \delta'$ with $\mathrm{APP}(\delta, h)$ and $\mathrm{APP}(\delta', h')$, but $h \neq h' \restriction \delta$. Let $\xi < \delta$ be least such that $h(\xi) \neq h'(\xi)$.

So, $h \restriction \xi = h' \restriction \xi$, and thus

$$h(\xi) = \mathbf{G}(h \restriction \xi) = \mathbf{G}(h' \restriction \xi) = h'(\xi)$$

which is impossible by the choice of ξ. $\lightning$

Existence

Existence is proved (using Uniqueness) by Transfinite Induction on **ON**:

$$\forall\delta\exists h\, \mathrm{APP}(\delta, h).$$

Suppose $\forall\xi < \delta\, \exists! h_\xi\, \mathrm{APP}(\xi, h_\xi)$.
There are three cases to consider, depending on δ:

① $\delta = 0$. Then $\mathrm{APP}(0, \emptyset)$.

② $\delta = S(\beta)$. Let $h = h_\beta \cup \{(\beta, \mathbf{G}(h_\beta))\}$. So,

(i) h is a function,

(ii) $\mathrm{dom}(h) = \delta$,

(iii) for all $\xi < \delta$, $h(\xi) = \mathbf{G}(h \restriction \xi)$:

$$\xi < \beta \implies h(\xi) = h_\beta(\xi) \text{ by } \mathrm{APP}(\beta, h_\beta);$$
$$\xi = \beta \implies h(\beta) = \mathbf{G}(h \restriction \beta) \text{ by definition.}$$

Finishing Proof

③ δ is a limit ordinal. By Uniqueness $\{h_\xi \mid \xi < \delta\}$ is a compatible family of functions, so define

$$h = \bigcup_{\xi<\delta} h_\xi$$

Note: Since δ is a limit ordinal: $\xi < \delta \implies S(\xi) < \delta$.

(i) h is a function (by HW3.)

(ii) $\mathrm{dom}(h) = \delta$:

$$\mathrm{dom}(h) = \bigcup_{\xi<\delta} \mathrm{dom}(h_\xi) = \bigcup_{\xi<\delta} \xi = \delta$$

Suppose $\xi < \delta$. Then $S(\xi) < \delta$ and so,
$\xi \in \mathrm{dom}(h_{S(\xi)}) \subseteq \mathrm{dom}(h)$ (by HW3).

(iii) For all $\xi < \delta$, $h(\xi) = \mathbf{G}(h \restriction \xi)$:
since $h(\xi) = h_{S(\xi)}(\xi)$, this follows by $\mathrm{APP}(S(\xi), h_{S(\xi)})$.