Math 582
Introduction to Set Theory

Kenneth Harris
kaharri@umich.edu

Department of Mathematics
University of Michigan

March 2, 2009

Outline

Introduction

☞ The next couple of weeks will cover ordinal numbers, transfinite induction and recursion, and ordinal arithmetic. These extend the themes we have developed with the natural numbers to the transfinite.

☞ The material is from Chapter 6 of Hrbacek and Jech (although I will do Section 6.1 last).

Natural numbers

☞ From now on we will take the natural numbers $\mathbb{N}$ to be the set ω:

$$\omega = \{ \overbrace{\emptyset}^{0}, \overbrace{\{\emptyset\}}^{1}, \overbrace{\{\emptyset, \{\emptyset\}\}}^{2}, \overbrace{\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}}^{3}, \dots \}$$

where successor is $S(n) = n \cup \{n\}$ and the natural order $<$ is $\in$.

☞ We established two important properties about the structure $(\omega, \in)$.

(a) ω is a transitive set: $n \subseteq \omega$, for each $n \in \omega$. The reason is that $n = \{0, 1, \dots, n-1\}$, the set of smaller numbers.

(b) ω is well-ordered by $\in$. This is just a restatement of the fact that any natural number system $\mathbb{N}$ is well-ordered by its natural ordering $<$.

Outline

Laying your bats in a row

To begin counting a collection $\mathcal{B}$ of 🦇 you need to get your bats in a row, then pair them off against the natural numbers until you run out of 🦇's:

🦇	🦇	🦇	🦇	🦇
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
0	1	2	3	4

We have paired the 🦇's against the natural number $5 = \{0, 1, 2, 3, 4\}$. Five, a cardinal number, is the answer to the question How many bats are there?

In the process of counting how many bats we have ordered the 🦇's:

① first 🦇
② second 🦇
③ third 🦇
④ fourth 🦇
⑤ fifth 🦇

Cardinal numbers and Ordinal numbers

☞ Cardinal numbers are answers to the question

How many objects are there in the set $\mathcal{B}$?

☞ Ordinal numbers are answers to the question

What position does this object occupy in some ordering?

We will use well-orderings (ordinals) to count objects because any two well-orderings are comparable.

☞ For finite number, cardinal and ordinal numbers are the same by the Pigeon-Hole Principle.

Transfinite number

☞ Cantor discovered the process of counting to the transfinite. For example, to count the rational numbers $\mathbb{Q}$ you only need ω:

$$\begin{array}{cccccccc} 0 & \frac{1}{1} & -\frac{1}{1} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{2} & -\frac{1}{2} & \dots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \dots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \end{array}$$

The cardinality of $\mathbb{Q}$ (written $|\mathbb{Q}|$) is $\aleph_0$, and we discovered this by ordering $\mathbb{Q}$ as ω.

(Not the usual order $<$ on $\mathbb{Q}$).

☞ More generally, a set is countable if it can be put into one-to-one correspondence with ω.

Real numbers and the transfinite

☞ Cantor's diagonal argument showed that every countable collection $X_\omega = \{r_0, r_1, r_2, \ldots, \}$ of real numbers misses a real number.

☞ There is no reason not to add this missed number to X_ω:

$$\begin{aligned} X_{\omega+1} &= \{r_0, r_1, \ldots, r_\omega\} && r_\omega \notin X_\omega \\ X_{\omega+2} &= \{r_0, r_1, \ldots, r_\omega, r_{\omega+1}\} && r_{\omega+1} \notin X_{\omega+1} \\ X_{\omega+3} &= \{r_0, r_1, \ldots, r_\omega, r_{\omega+1}, r_{\omega+2}\} && r_{\omega+2} \notin X_{\omega+2} \end{aligned}$$

☞ We continue counting beyond ω:

$$0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \omega+3, \ldots, \omega+\omega, \ldots$$

but we still have countably many objects in this extended ordering of ω:

$$\begin{array}{ccccccccc} 0 & \omega & 1 & \omega+1 & 2 & \omega+2 & 3 & \omega+3 & \ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \ldots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \end{array}$$

Initial segment of the ordinal numbers

☞ We can continue counting into the transfinite:

$$0, 1, 2, \ldots, \omega, \omega{+}1, \omega{+}2, \ldots, \omega{\cdot}2, \ldots, \omega{\cdot}3, \ldots, \omega{\cdot}\omega, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \omega_1, \ldots$$

Eventually we reach the first ordinal number after ω which has a greater cardinality than ω; this uncountable ordinal is ω_1 and its cardinality is $\aleph_1$.

The Continuum Hypothesis is the question whether the cardinality of $\mathbb{R}$ is $\aleph_1$. Equivalently, can we order $\mathbb{R}$ as ω_1 in such a way that we exhaust all reals?

The idea of counting into the transfinite is due to Cantor who first raised and investigated the Continuum Hypothesis. Our definition of the ordinals is due to von Neumann.

Outline

Transitive sets

Definition

A set z is a transitive set if $\forall y \in z\,(y \subseteq z)$

- We defined each natural number $n = \{0, 1, 2 \ldots, n-1\}$ and proved that $\in$ is transitive on these number.
- $\omega = \{0, 1, 2, 3, \ldots,\}$ is a transitive set.
- Nontransitive sets: $\{1, 2, 3\}$. Note that this set is order-isomorphic to the transitive set $3 = \{0, 1, 2\}$. So, being a transitive set is different from being transitive under the ordering $\in$.
- If z is a transitive set, then $\in$ is a transitive relation on $z \cup \bigcup z$:

$$\forall x, y\,\big(x \in y \wedge y \in z \quad \rightarrow \quad x \in z\big)$$

Ordinals defined

The following definition of ordinal is due to von Neumann (mid-20's):

Definition
An ordinal is a transitive set, well-ordered by ϵ.

- Some ordinals: $0, 1, 2, 3, 4 \ldots$.
- If x is an ordinal, then so is $S(x) = x \cup \{x\}$ (we will verify this shortly.)
- $\omega = \{0, 1, 2, 3, \ldots, \}$ is an ordinal. (We will prove this soon.)
- If X is well-ordered by $\in$, then $\in \restriction X \times X$ is a transitive relation. However, X may not be a transitive set:

  $X = \{1, 2, 3\}$ is well-ordered by $\in$, but not a transitive set.

Convention

Convention. Standard practice is to use α, β, γ (lowercase Greek letters) to denote ordinals. We also use the following abbreviations:

- $\alpha < \beta$ means $\alpha \in \beta$,
- $\alpha \leq \beta$ means $\alpha < \beta \vee \alpha = \beta$,
- $\forall \alpha\, \varphi(\alpha)$ abbreviates

  $$\forall x\, (x \text{ is an ordinal} \rightarrow \varphi(x))$$

Notation

Informally, we define the class of all ordinals:

$$\text{ON} = \{x \mid x \text{ is an ordinal }\}$$

although, we will shortly show that it is a proper class (so, not a set – this is the Burali-Forti paradox.)

The following notation is still useful to have:

- $x \in \text{ON}$ abbreviates "x is an ordinal"
- $x \subseteq \text{ON}$ abbreviates "$\forall y \in x\,(y \text{ is an ordinal })$"
- $x \cap \text{ON}$ abbreviates "$\{y \in x \mid y \text{ is an ordinal }\}$"

ON is well-ordered by $\in$

Theorem

ON is well-ordered by $\in$. That is,

1. *$\in$ is transitive on the ordinals: $\forall \alpha, \beta, \gamma\,(\alpha < \beta \wedge \beta < \gamma \;\rightarrow\; \alpha < \gamma)$.*
2. *$\in$ is irreflexive on the ordinals: $\forall \alpha\, \alpha \not< \alpha$.*
3. *$\in$ satisfies trichotomy on the ordinals: $\forall \alpha, \beta\,(\alpha < \beta \vee \beta < \alpha \vee \alpha = \beta)$.*
4. *$\in$ is well-founded on the ordinals: Every nonempty set of ordinals has an $\in$-least member.*

Three little lemmas

We will need to prove three little lemmas.

☺ Sanity check: verify these hold for the natural numbers.

Lemma (Lemma 1)

ON is a transitive class: that is, if $\alpha \in$ ON then $\alpha \subseteq$ ON.

Lemma (Lemma 2)

$\alpha \cap \beta$ is an ordinal, when α and β are ordinals.

Lemma (Lemma 3)

For all ordinals α and β, $\alpha \subseteq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$

Proof of Theorem from Three Lemmas

☞ Transitivity: Restates fact that γ is a transitive set:

$$\forall \alpha, \beta, \gamma \left(\alpha \in \beta \wedge \beta \in \gamma \ \rightarrow \ \alpha \in \gamma\right).$$

☞ Irreflexivity: Restates fact that α is irreflexive:

☛ $x \notin x$ for all $x \in \alpha$, so in particular, $\alpha \notin \alpha$.

Proof of Theorem from Three Lemmas

☞ Trichotomy: Fix α, β and let $\delta = \alpha \cap \beta$, so that δ is an ordinal by Lemma 2. But by Lemma 3,

(i) $\delta \subseteq \alpha$, so $\delta \in \alpha$ or $\delta = \alpha$.

(ii) $\delta \subseteq \beta$, so $\delta \in \beta$ or $\delta = \beta$.

But, we cannot have both $\delta \in \alpha$ and $\delta \in \beta$:

☛ otherwise, $\delta \in \alpha \cap \beta = \delta$, contradicting irreflexivity.

Thus, either $\delta = \alpha$ or $\delta = \beta$.

☞ well-founded: Let X be a nonempty set of ordinals, and fix $\alpha \in X$.

☛ If $\alpha \cap X = \emptyset$ then α is $\in$-least from X;

☛ otherwise, $\alpha \cap X \neq \emptyset$ so has an $\in$-least element in α.

This element is $\in$-least in ON, since α is a transitive set.

Burali-Forti

Our definition of ordinal is due to von Neumann, some thirty years after Cesare Burali-Forti stated this theorem.

Theorem (Burali-Forti)

ON is a proper class; that is, there is no set containing all ordinals.

Proof.

Suppose there is a set X containing all ordinals.

By Comprehension,

$$ON = \{y \in X \mid y \text{ is an ordinal}\}$$

ON is a transitive class (by Lemma 1) and is well-ordered by $\in$; so, ON is an ordinal, and thus ON $\in$ ON.

This contradicts irreflexivity. ↯ □

Outline

Lemma 1

Lemma (Lemma 1)

ON is a transitive class: that is, if $\alpha \in ON$ then $\alpha \subseteq ON$.

Proof.

Suppose $\alpha \in$ ON and $z \in \alpha$, so we need to show $z \in$ ON.

(Well-Ordering). Since $z \subseteq \alpha$ it follows that $\in$ well-orders z.

(Transitive Set). Suppose $x \in y$ and $y \in z$, and we need to show $x \in z$.

☛ Since $y \in z \subseteq \alpha$, it follows that $y \subseteq \alpha$; so, $x \in \alpha$.

☛ We now have $x, y, z \in \alpha$; but, $\in$ is a transitive relation on α, so from $x \in y$ and $y \in z$ it follows that $x \in z$. □

Lemma 2

☞ $\alpha \cap \beta$ is the smaller of α and β by Trichotomy. But we needed Lemma 2 below to prove this.

Note. If x, y are transitive sets, then so its $x \cap y$.

Lemma (Lemma 2)

$\alpha \cap \beta$ is an ordinal, whenever both α and β are ordinals.

Proof.

Since $\alpha \cap \beta \subseteq \alpha$, it is well-ordered by $\in$, and is a transitive set by the above **Note**.

□

Lemma 3

Lemma (Lemma 3)

For all ordinals α and β, $\alpha \subseteq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$

Proof.

($\Leftarrow$). β is a transitive set , so $\alpha \in \beta$ implies $\alpha \subseteq \beta$.

($\Rightarrow$). Assume $\alpha \subsetneq \beta$. We will show $\alpha \in \beta$.
Let $X = \beta - \alpha$ and ξ $\in$-least in X, and thus, $\xi \notin \alpha$. We show $\xi = \alpha$.

☞ $\xi \subseteq \alpha$: If $\mu \in \xi$, then $\mu \notin \beta - X$ (ξ is $\in$-least in X), so $\mu \in \alpha$.

☞ $\alpha \subseteq \xi$: Let $\mu \in \alpha$, so that $\mu, \xi \in \beta$. But β satisfies trichotomy so one of $\mu = \xi$ or $\xi \in \mu$ or $\mu \in \xi$.

- ☛ Since $\xi \notin \alpha$ but $\mu \in \alpha$ we cannot have $\xi = \mu$.
- ☛ Since $\xi \notin \alpha$ and α is transitive, we cannot have $\xi \in \mu \in \alpha$.

Therefore, $\mu \in \xi$.

✓ Therefore, $\xi = \alpha$, so that $\alpha \in \beta$.

□

Outline

Successor Ordinals

Recall. $S(x) = x \cup \{x\}$

Lemma

If α is an ordinal, then so is $S(\alpha)$;
furthermore, for all ordinals γ, $\gamma < S(\alpha)$ iff $\gamma \leq \alpha$.

Proof.

(**First part**). $\alpha \cup \{\alpha\}$ is a transitive set, well-ordered by $\in$ (with α as the $\in$-greatest element.)

(**Second part**). $\gamma \in \alpha \cup \{\alpha\}$ iff $\gamma \in \alpha$ or $\gamma = \alpha$.

□

Types of ordinals

Definition

An ordinal β is

- a successor ordinal if $\beta = S(\alpha)$ for some α,
- a limit ordinal if $\beta \neq 0$ and not a successor ordinal,
- a finite ordinal if every $\alpha \leq \beta$ is either 0 or a successor ordinal.

Recall, x is a natural number if $x \in \omega$.

Lemma

If x is a finite ordinal then so is every $y \in x$, and $S(x)$ is also a finite ordinal. Furthermore, every natural number is a finite ordinal.

Proof. HW4.

ω is the set of finite ordinals

Lemma

ω is the set of finite ordinals.

Proof.

Suppose x is a finite ordinal but $x \notin \omega$, so $Y = S(x) - \omega \neq \emptyset$.
Let $z \in S(x) - \omega$ be $\in$-least.

z is a finite ordinal and $z \neq 0$ (since $0 \in \omega$), so $z = S(w)$ for some finite ordinal w.

- Since $w \in z \in S(x)$, we have $w \in S(x)$; so, $w \in \omega$.
- Thus, $z = S(w) \in \omega$ (recall, ω is an inductive set.) ↯

✓ Thus, $x \in \omega$. □

ω an ordinal

We now show that ω is an ordinal.

Lemma

Assume $X \subseteq ON$ and is an initial segment of ON:

$$\forall \beta \in X\, \forall \alpha < \beta\, \alpha \in X$$

Then $X \in ON$.

Proof.

X is well-ordered by $\in$ since $X \subseteq$ ON.
That X is an initial segment is just that X is a transitive set. □

Note. The set of finite ordinals (that is, ω) an an initial segment of ON.

ω an ordinal

Lemma

ω is the least limit ordinal.

Proof.

- ☞ ω is an initial segment of ordinals (it is the set of finite ordinals, which form an initial segment.)
  Thus, ω is an ordinal.
- ☞ ω cannot be itself a successor since it is not a finite ordinal.
  Thus, ω is a limit ordinal.
- ☞ Every member of ω is a finite ordinal, so is either 0 or a successor.
  Thus, ω is the least limit ordinal. □