Math 582
Introduction to Set Theory

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Induction Scheme

We will be working with the concept of a system of natural numbers, as given by the Dedekind-Peano axioms. There are two main results in this lecture:

1. The (Primitive) Recursion Theorem, the main theorem used for constructing functions on the domain of a system of natural numbers.
2. The Uniqueness of the system of natural numbers, all systems of natural numbers are isomorphic.

Induction Scheme

Note. The formulas φ in the statement of the theorem are formulas in set theory.

Theorem

Let $(\mathbb{N}, 0, S)$ *be a system of natural numbers. Let* $\varphi(x)$ *be any formula. Suppose*

(a) $\varphi(0)$ *is true. (basis case)*

(b) $\forall n \in \mathbb{N}\left(\varphi(n) \rightarrow \varphi(S(n))\right)$ *is true. (inductive case)*

Then $\forall n \in \mathbb{N}\, \varphi(n)$*.*

Proof.

By Comprehension, let $A = \{n \in \mathbb{N} \mid \varphi(n)\}$.
From (a) $0 \in A$ and from (b), $n \in A \rightarrow S(n) \in A$ for all $n \in \mathbb{N}$.
Then, by the Induction Principle (**N5**), $A = \mathbb{N}$, i.e. $\forall n \in \mathbb{N}\, \varphi(n)$. □

Example of Induction Scheme

Lemma

Let $(\mathbb{N}, 0, S)$ *be a system of natural numbers. Every nonzero element is a successor:*

$$\forall n \in \mathbb{N}\left(n \neq 0 \rightarrow \exists m \in \mathbb{N}(n = S(m))\right)$$

Proof.

Proof by Induction using the formula

$$\varphi(n) \quad := \quad n \neq 0 \rightarrow \exists m \in \mathbb{N}(n = S(m))$$

☞ basis. $0 \neq 0$ is false, so $\varphi(0)$ is true.

☞ inductive. Suppose $\varphi(n)$ (i.h., inductive hypothesis), show $\varphi(S(n))$.
But, the consequent of $\varphi(S(n))$ is true, so $\varphi(S(n))$ is true.

✓ Therefore, $\forall n \in \mathbb{N}\, \varphi(n)$. □

Example of Induction Scheme

Lemma

Let $(\mathbb{N}, 0, S)$ be a system of natural numbers. Then $n \neq S(n)$ for every $n \in \mathbb{N}$.

Proof.

Proof by Induction.

☞ basis. $0 \neq S(0)$ by (**N4**).

☞ inductive. Suppose $n \neq S(n)$ (i.h.), show $S(n) \neq S(S(n))$.
Suppose $S(n) = S(S(n))$. Then, $n = S(n)$ by (**N3**). ↯

Thus, $S(n) \neq S(S(n))$.

✓ Therefore, $n \neq S(n)$ for every $n \in \mathbb{N}$.

□

Constructing bijections on natural number systems

Let $(\mathbb{N}_1, 0_1, S_1)$ and $(\mathbb{N}_2, 0_2, S_2)$ be systems of natural numbers.

Here is how you could compute a bijection $\pi : \mathbb{N}_1 \to \mathbb{N}_2$:

$$\begin{array}{rcl} 0_1 & \xrightarrow{\pi} & 0_2 \\ S_1(0_1) & \xrightarrow{\pi} & S_2(0_2) \\ S_1(S_1(0_1)) & \xrightarrow{\pi} & S_2(S_2(0_2)) \\ S_1(S_1(S_1(0_1))) & \xrightarrow{\pi} & S_2(S_2(S_2(0_2))) \\ & \vdots & \end{array}$$

Iterate the operator S_2:

$$n \xrightarrow{\pi} S_2^{(n)}(0_2)$$

where $n = S_1^{(n)}(0_1)$.

Primitive Recursion

Theorem (Primitive Recursion Theorem – basic version)

Let $(\mathbb{N}, 0, S)$ be a system of natural numbers, E a set, $a \in E$ and $h : E \to E$. Then there is a unique function $F : \mathbb{N} \to E$ satisfying for every $n \in \mathbb{N}$

$$\begin{aligned} f(0) &= a \\ f(S(n)) &= h(f(n)) \end{aligned}$$

Examples

- $f : \mathbb{N}_1 \to \mathbb{N}_2 : a = 0_2$, $h : x \mapsto S_2(x)$, where $(\mathbb{N}_2, 0_2, S_2)$ is a system of natural numbers.
- $f(n) = 5n : a = 0$, $h : x \mapsto 5 + x$.
- $f(n) = b^n : a = 1$, $h : x \mapsto b \cdot x$.
- $f(n) = h^{(n)}(a) : a = a$, $h : x \mapsto h(x)$.

Proof of Recursion Theorem

Let E be a set, $a \in E$ and $h : E \to E$.

Define a predicate $\texttt{COMP}(t)$ ("t is a *computation*") if and only if

- $t \subseteq \mathbb{N} \times E$ and t is a function,
- $0 \in \text{dom}(t)$ and $t(0) = a$,
- $\forall n \in \mathbb{N}\,(S(n) \in \text{dom}(t) \;\to\; n \in \text{dom}(t) \wedge t(S(n)) = h(t(n)))$

Idea of Proof.

☞ Show that if $\texttt{COMP}(t)$ and $\texttt{COMP}(u)$ then t and u are compatible functions (see Exercise 4 off HW5.)

☞☞ Define $f = \bigcup\{t \mid \texttt{COMP}(t)\}$. (This is a function by Exercise 4 of HW5.)

Step ☞

Lemma (Step ☞)
If $\mathrm{COMP}(t)$ *and* $\mathrm{COMP}(u)$ *then t and u are compatible:*

$$\forall n \in \mathbb{N}\,(n \in dom(t) \cap dom(u) \;\rightarrow\; t(n) = u(n)).$$

Proof. By Induction. Let

$$X \;:=\; \{n \in \mathbb{N} \,|\, \forall t, u(\mathrm{COMP}(t) \wedge \mathrm{COMP}(u) \wedge n \in \mathrm{dom}(t) \cap \mathrm{dom}(u) \;\rightarrow\; t(n) = u(n))\}$$

☞ basis. $0 \in X$ by definition of COMP.

Proof of Step ☞

☞ induction. Suppose $n \in X$ (i.h.), and that the antecedent of our condition holds for $S(n)$ for some arbitrary t, u:

$$\mathrm{COMP}(t) \wedge \mathrm{COMP}(u) \wedge S(n) \in \mathrm{dom}(t) \cap \mathrm{dom}(u).$$

Then $n \in \mathrm{dom}(t) \cap \mathrm{dom}(u)$ (by COMP) and so, $t(n) = s(n)$ (by i.h.). Thus,

$$t(S(n)) = h(t(n)) = h(u(n)) = u(S(n)).$$

So, $S(n) \in X$.

✓ Therefore, $X = \mathbb{N}$.

Step ☞☞

We can define $f = \bigcup\{t \mid \text{COMP}(t)\}$; but we must be sure $\text{dom}(f) = \mathbb{N}$.

Lemma (Step ☞☞)

For each $n \in \mathbb{N}$ there is a t with $\text{COMP}(t)$ and $n \in \text{dom}(t)$.

Proof.

By induction. Let

$$X \quad := \quad \{n \in \mathbb{N} \mid \exists t\,(\text{COMP}(t) \wedge n \in \text{dom}(t))\}$$

☞ basis. Since $\{(0, a)\}$ is a computation, $0 \in X$.

☞ inductive. Suppose $n \in X$. Let t be given from i.h.
If $S(n) \in t$, we are done. Otherwise, let $u = t \cup \{(S(n), h(t(n)))\}$.

Verify (✍) $\text{COMP}(u)$. So, $S(n) \in X$.

✓ $X = \mathbb{N}$.

□

Completing proof

We have shown from Step ☞ and Step ☞☞:

✵ For every $n \in \mathbb{N}$ there is a unique $w \in E$ such that $(n, w) \in t$, for any computation t with $n \in \text{dom}(t)$.

We can define (using Comprehension)

$$f \quad := \quad \{(n, w) \in \mathbb{N} \times E \mid \exists t\,(\text{COMP}(t) \wedge n \in \text{dom}(t) \wedge (n, w) \in t)\},$$

so that by (✵), $f : \mathbb{N} \to E$. We still need to show that f satisfies the conditions (for every $n \in \mathbb{N}$):

$$\begin{aligned} f(0) &= a \\ f(S(n)) &= h(f(n)) \end{aligned}$$

This an easy induction ☺.

Uniqueness Theorem

Theorem

For any two systems of natural numbers $(\mathbb{N}_1, 0_1, S_1)$ *and* $(\mathbb{N}_2, 0_2, S_2)$, *there exists a unique bijection* $\pi : \mathbb{N}_1 \rightleftarrows \mathbb{N}_2$ *satisfying the following (structure preserving) properties:*

$$\begin{aligned} \pi(0_1) &= 0_2 \\ \pi(S_1(n)) &= S_2(\pi(n)) \end{aligned}$$

We call π *the canonical isomorphism from* $(\mathbb{N}_1, 0_1, S_1)$ *onto* $(\mathbb{N}_2, 0_2, S_2)$, *the say the two systems are isomorphic.*

Proof of Uniqueness

Let $(\mathbb{N}_1, 0_1, S_1)$ and $(\mathbb{N}_2, 0_2, S_2)$ be two systems of natural numbers. Use the Primitive Recursion Theorem, with $E = \mathbb{N}_2$, $a = 0_2$ and $h = S_2$. Then there exists a unique map $\pi : \mathbb{N}_1 \to \mathbb{N}_2$ satisfying

$$\begin{aligned} \pi(0_1) &= 0_2 \\ \pi(S_1(n)) &= S_2(\pi(n)) \end{aligned}$$

We must prove

- ☞ π is surjective.
- ☞☞ π is injective.

π is surjective

Step ✍. π is surjective.

We prove π is surjective by induction on $\mathbb{N}_2$, by showing $\pi[\mathbb{N}_1] = \mathbb{N}_2$.

☞ $\pi(0_1) = 0_2$, so $0_2 \in \pi[\mathbb{N}_1]$.

☞ Suppose $n \in \pi[\mathbb{N}_1]$; then there is an $m \in \mathbb{N}_1$ with $\pi(m) = n$. So,

$$\pi(S_1(m)) = S_2(\pi(m)) = S_2(n).$$

Thus, $S_2(n) \in \pi[\mathbb{N}_1]$.

✓ Therefore, $\pi[\mathbb{N}_1] = \mathbb{N}_2$, and π is surjective.

π is injective

Step ✍✍. We prove π is injective by induction on $\mathbb{N}_1$. Let

$$X = \{n \in \mathbb{N}_1 \,|\, \forall m \in \mathbb{N}_1 \left(\pi(m) = \pi(n) \;\rightarrow\; m = n\right)\}$$

The induction is on n.

☞ basis. $n = 0_1$. Then $\pi(0_1) = 0_2$.
If $m \neq 0_1$, then $m = S_1(k)$ for some k. So,

$$\pi(m) = \pi(S_1(k)) = S_2(\pi(k)),$$

and so $\pi(m) \neq 0_2 = \pi(0_1)$ (by condition (**N4**).) Thus, $0_1 \in X$.

π is injective–inductive step

$$X = \{n \in \mathbb{N}_1 \mid \forall m \in \mathbb{N}_1 \, (\pi(m) = \pi(n) \ \rightarrow \ m = n)\}$$

☞ inductive. Suppose $n \in X$ (i.h.), and that $\pi(S_1(n)) = \pi(m)$. Since

$$\pi(m) = \pi(S_1(n)) = S_2(\pi(n)),$$

it follows that $\pi(m) \neq 0_2$; and so, $m \neq 0_1$ (since $\pi(0_1) = 0_2$.)
Let $m = S_1(k)$; then

$$S_2(\pi(n)) = \pi(m) = \pi(S_1(k)) = S_2(\pi(k)).$$

So, $\pi(n) = \pi(k)$ by (**N3**), and $n = k$ by (i.h.).
Therefore, $S_1(n) = S_1(k) = m$. Since m was arbitrary, $S_1(n) \in X$.

✓ $X = \mathbb{N}_1$, and so π is injective.