Math 582
Introduction to Set Theory

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Dedekind-Peano Axioms

Definition (Dedekind (1888), Peano (1892))

A system of natural numbers is a triple $(\mathbb{N}, 0, S)$ satisfying:

N1 $0 \in \mathbb{N}$

N2 $S : \mathbb{N} \to \mathbb{N}$

N3 $\forall n, m\big[S(n) = S(m) \to n = m\big]$ (i.e. S is injective),

N4 $\forall n\, S(n) \neq 0$

N5 **Induction Principle.** For every $X \subseteq \mathbb{N}$,

$$0 \in X \wedge \forall n \in \mathbb{N}(n \in X \to S(n) \in X) \quad \to \quad X = \mathbb{N}$$

Examples.

- The natural numbers: $(\mathbb{N}, 0, s)$ where $s(n) = n + 1$.
- The odd numbers: $(\mathbb{O}, 1, d)$ where $d(n) = n + 2$.

Goals

A. **Existence** of a system of natural numbers (which will require a new axiom.)

B. **Uniqueness** of systems of natural numbers (where uniqueness means the existence of a structure preserving bijection – to be explained below.)

C. **Well-ordered** by a natural ordering on any system of natural numbers.

D. **Basic operations** (addition, multiplication, exponentiation) are definable on any system of natural numbers in a natural way.

Existence Theorem

We will need a new axiom, Axiom of Infinity, to prove that there exists a system of natural numbers.

Theorem

There exists at least one system of natural numbers $(\mathbb{N}, 0, S)$.

Uniqueness Theorem

Theorem

For any two systems of natural numbers $(\mathbb{N}_1, 0_1, S_1)$ *and* $(\mathbb{N}_2, 0_2, S_2)$*, there exists a unique bijection* $\pi : \mathbb{N}_1 \rightleftarrows \mathbb{N}_2$ *satisfying the following (structure preserving) properties:*

$$\begin{aligned} \pi(0_1) &= 0_2 \\ \pi(S_1(n)) &= S_2(\pi(n)) \end{aligned}$$

We call π *the canonical isomorphism from* $(\mathbb{N}_1, 0_1, S_1)$ *onto* $(\mathbb{N}_2, 0_2, S_2)$*, the say the two systems are isomorphic.*

Well-ordering on systems of natural numbers

We are going to show how to define a relation $<$ for any system of natural numbers $(\mathbb{N}, 0, S)$, the canonical ordering on $\mathbb{N}$, so that the ordered set $(\mathbb{N}, <)$ is a well-ordered set. This ordering will be natural in the sense that we will be able to prove the following theorem:

Theorem

Let $(\mathbb{N}_1, 0_1, S_1)$ *and* $(\mathbb{N}_2, 0_2, S_2)$ *be two systems of natural numbers, where* $<_1, <_2$ *are their respective canonical well-orders. Then the canonical isomorphism* $\pi : \mathbb{N}_1 \rightleftarrows \mathbb{N}_2$ *is order preserving:*
for all $n, m \in \mathbb{N}_1$*:*

$$n <_1 m \leftrightarrow \pi(n) <_2 \pi(m).$$

Canonical operations on systems of natural numbers

We are going to show how to define canonical operations of $+$ (addition) and $\cdot$ (multiplication) on any natural number system $(\mathbb{N}, 0, S)$ so that these operations agree with our "common sense" understanding of addition and multiplication.

These will also be natural in the following sense:

Theorem

Suppose $(\mathbb{N}_1, 0_1, S_1)$ *and* $(\mathbb{N}_2, 0_2, S_2)$ *are two systems of natural numbers, where* $+_1, \cdot_1, +_2, \cdot_2$ *are their respective canonical operations of addition and multiplication. Then the canonical isomorphism* $\pi : \mathbb{N}_1 \rightleftarrows \mathbb{N}_2$ *respects these operations: for all* $n, m \in \mathbb{N}_1$*:*

$$\begin{aligned} \pi(n +_1 m) &= \pi(n) +_2 \pi(m) \\ \pi(n \cdot_1 m) &= \pi(n) \cdot_2 \pi(m) \end{aligned}$$

Defining "natural number" in set theory

Definition. The ordinal successor function is defined by $S(x) = x \cup \{x\}$ for any set x.

$$0 = \emptyset \quad 1 = S(0) = \{0\} \quad 2 = S(1) = \{0, 1\} \quad 3 = S(2) = \{0, 1, 2\} \quad \ldots$$

We would like to "define" a system of natural numbers $(\omega, 0, S)$ by

(a) $0 \in \omega$

(b) If $n \in \omega$ then $S(n) \in \omega$

(c) All members of ω are obtained by application of (a) and (b).

This is an example of an inductive definition. The challenge is to capture property (c).

Inductive sets

Definition. A set I is called inductive if

(a) $0 \in I$

(b) If $x \in I$ then $S(x) \in I$

We define ω as the smallest inductive set:

$$\omega = \bigcap \{ I \mid I \text{ is inductive } \} = \{ x \mid \forall I (I \text{ is inductive } \rightarrow x \in I) \}$$

The problem is that this can only be justified when there is some inductive set I. (Otherwise, $\bigcap \emptyset = V$.) Our axioms so far do not justify the existence of inductive sets because these sets must be infinite, and our axioms do not justify the existence of any infinite set.

Axiom of Infinity

Axiom 7: Infinity:

$$\exists x \big(\emptyset \in x \ \wedge \ \forall y \in x (S(y) \in x) \big)$$

(i.e. there exists an inductive set.)

Definition

Let A be the inductive set given by Axiom 7. Then

$$\omega = \{ x \in A \mid \forall I (I \text{ is inductive } \rightarrow x \in I) \}$$

ω is the smallest inductive set

$$n \in \omega \quad \longleftrightarrow \quad \forall \text{ inductive } I \, (n \in I)$$

Lemma

ω is inductive; and, if I is any inductive set then $\omega \subseteq I$.

Proof.

- ☞ $0 \in \omega$. $0 \in I$ for every inductive set I by (a).
- ☞ Suppose $n \in \omega$ and show $S(n) \in \omega$. So, $n \in I$ for every inductive set, and thus $S(n) \in I$ for every inductive set I by (b). Therefore, $S(n) \in \omega$.

✓ ω is inductive.

The second half of the theorem follows from the definition of ω.

□

Existence of a system of natural numbers

A system of natural numbers

Let $\overline{S} = \{(n, m) \in \omega \times \omega \,|\, m = S(n)\}$.

Theorem

$(\omega, \emptyset, \overline{S})$ is a system of natural numbers.

Proof.

N1 $\emptyset \in \omega$. True since ω is inductive (a).

N2 $\overline{S} : \omega \to \omega$. Clear from definition.

N3 If $\overline{S}(n) = \overline{S}(m)$ then $n = m$. (Requires some work !!)

N4 $\overline{S}(n) \neq \emptyset$ for any n. Clear, since $n \in \overline{S}(n) = n \cup \{n\}$.

N5 **Induction Principle**. Suppose $X \subseteq \omega$ and X satisfies (a) $\emptyset \in X$ and (b') if $n \in X$ then $\overline{S}(n) \in X$. Show X is inductive.
(b') implies (b) if $n \in X$ then $S(n) \in X$; so X is inductive, and $\omega \subseteq X$. Thus, $X = \omega$.

□

Proving condition N3

If $\overline{S}(n) = \overline{S}(m)$ then $n = m$.

Proof.

Suppose $\overline{S}(n) = \overline{S}(m)$ but $n \neq m$. Since

$$\overline{S}(n) = n \cup \{n\} \quad \overline{S}(m) = m \cup \{m\},$$

so, $n \in m$ and $m \in n$. This violates the Axiom of Foundation:
let $x = \{m, n\}$, then

$$n \in x \cap m \ \wedge \ m \in x \cap n.$$

Therefore, $n = m$. □

☞ However, I said the Axiom of Foundation was never needed for mathematics.

Proving condition N3

Notice the following property of the first few members of ω.

- $0 = \emptyset \in \{\emptyset\} = 1$ and $0 \subseteq 1$,
- $0, 1 \in 2 = \{0, 1\}$ and $0, 1 \subseteq 2$,
- $0, 1, 2 \in 3 = \{0, 1, 2\}$ and $0, 1, 2 \subseteq 3$

This property holds generally,

Lemma (Transitivity of $\in$)

For any $n, m \in \omega$, if $m \in n$ then $m \subseteq n$.

The Lemma says: $\in$ is transitive on each $n \in \omega$:

$$\forall k, m \in \omega \left(k \in m \wedge m \in n \quad \rightarrow \quad k \in n\right).$$

Proof of Lemma

Proof.

Let $X \subseteq \omega$ be defined by

$$X = \{n \in \omega \,|\, \forall m\, [m \in n \;\rightarrow\; m \subseteq n])\}$$

It is sufficient to show X is inductive, which implies $X = \omega$.

☞ Show $0 \in X$. $0 = \emptyset$, so the antecedent condition is always false.

☞ Suppose $n \in X$ and show $S(n) \in X$. Notice that $n \subseteq n \cup \{n\} = S(n)$. Let $m \in S(n)$, so $m \in n$ or $m = n$.

$$\begin{aligned} m = n &\rightarrow m \subseteq S(n) \\ m \in n &\rightarrow m \subseteq n \subseteq S(n) \qquad \text{since } n \in X \end{aligned}$$

Thus, $m \subseteq S(n)$. So, $S(n) \in X$.

✓ So, X is inductive, and $X = \omega$.

□

Proving condition N3

We could use Foundation to prove the next property, but it is not necessary to use this axiom, in the special case of the members of ω.

Lemma (Irreflexivity of $\in$)

$n \notin n$ for every $n \in \omega$.

Proof of Lemma

Proof.

Let $X \subseteq \omega$ be defined by

$$X = \{n \in \omega \,|\, n \notin n)\}$$

It is sufficient to show X is inductive, which implies $X = \omega$.

☞ Show $0 \in X$. $0 = \emptyset$, and $\emptyset \notin \emptyset$.

☞ Suppose $n \in X$ and show $S(n) \in X$. Suppose $S(n) \in S(n)$. Then either $S(n) \in n$ or $S(n) = n$. If $S(n) \in n$ then $S(n) \subseteq n$ (by transitivity of $\in$), so in either case $S(n) \subseteq n$.

But, $n \in S(n) \subseteq n$, so $n \in n$ which contradicts $n \in X$. ↯

Thus, $S(n) \notin S(n)$, so $S(n) \in X$.

✓ So, X is inductive, and $X = \omega$. □

Proving condition N3

If $\overline{S}(n) = \overline{S}(m)$ then $n = m$.

Proof.

Suppose $\overline{S}(n) = \overline{S}(m)$ but $n \neq m$. Since

$$\overline{S}(n) = n \cup \{n\} \quad \overline{S}(m) = m \cup \{m\}$$

so, $n \in m$ and $m \in n$. Thus, $n \in n$ (by transitivity of $\in$) which contradicts the irreflexivity of $\in$. ↯

✓ Therefore, $n = m$.

□

Therefore, $(\omega, \emptyset, \overline{S})$ is a system of natural numbers.

The natural numbers

✍ From now on we write: $(\omega, 0, S)$ for this specific instance of a system of natural numbers.

✍ We write $(\mathbb{N}, 0, S)$ when we are talking about any system satisfying the Dedekind-Peano axioms.