Math 582
Introduction to Set Theory

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Relation to Text

This material is mainly from Section 2.5 of Hrbacek and Jech.
(Well-orders are introduced in Chapter 3, Definition 2.3 on p. 44. They form the backbone for our development of the ordinal numbers.)

Properties of relations

Let R be a relation, and A a set.

- R is transitive on A iff $\forall x, y, z \in A(xRy \wedge yRz \quad \rightarrow \quad xRz)$.
- R is reflexive on A iff $\forall x \in A(xRx)$.
- R is irreflexive on A iff $\forall x \in A(x \not R x)$.
- R is symmetric on A iff $\forall x, y \in A(xRy \rightarrow yRx)$.
- R is antisymmetric on A iff $\forall x, y \in A(xRy \wedge yRx \rightarrow x = y)$
- R satisfies trichotomy on A iff $\forall x, y \in A(xRy \vee yRx \vee x = y)$.

Types of relations: Equivalence

☞ Binary relations of several special types arise frequently.

Let R be a relation, and A a set.

R is an equivalence relation on A iff R is reflexivity, symmetric and transitivity on A. (See section 2.4 of H+J for more.)

Examples. The following are equivalence relations.

- The identity relation on a set A: $\{(a, a) \mid a \in A\}$.
  This is a set by Comprehension.
- The following relation $\sim$ on $\mathbb{N} \times \mathbb{N}$:

$$(n, m) \sim (p, q) \leftrightarrow nq = pm$$

Types of relations: Orders

☞ Another prominent type of binary relation is an order.
Let R be a relation, and A a set.

There are two main types of orders:

- R partially orders A iff R is reflexive, transitive and antisymmetric.
- R totally orders A iff R partially orders A and satisfies trichotomy. (Also, called a linear order.)

Orders come in a strict form:

- R partially orders A strictly iff R is transitive and irreflexive on A.
- R totally orders A strictly iff R partially orders A strictly and satisfies trichotomy on A.

Ordered sets

Definition

A pair (A, R) is a (strictly) ordered set if R is a relation which partially orders A (strictly.) If R totally orders A (strictly) then (A, R) is a (strict) totally ordered set.

Examples.

- $(\mathbb{N}, <)$ is a strict totally ordered set; $(\mathbb{N}, \leq)$ is a totally ordered set.
- $(\mathcal{P}(A), \subset)$ is a strict partially ordered set; $(\mathcal{P}(A), \subseteq)$ is a partially ordered set.
- $(A, \in)$ is not generally an ordered set.

Convention for ordered sets

Convention. We will be mostly studying irreflexive orders, strict (partial/total) orders. (This is in contrast with ordinary mathematics where the orders are usually take to be reflexive.) We will use the symbols $<, \prec, \lhd$ to denote strict orders (either partial or total.)

☞ We will use the symbols $\leq, \preceq, \unlhd$ to denote partial or total order. The reflexive relation corresponding to the strict order is:
$x \leq y \;\leftrightarrow\; x < y \vee x = y$ and $x \preceq y \;\leftrightarrow\; x \prec y \vee x = y$.

☞ We will call a pair $(A, <)$ (correspondingly, $\prec, \lhd$) simply orders, where it is understood that they are strict.

Example

$\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$ are not official sets, yet. Informally,

- $(\mathbb{Q}, <)$ is a strict totally ordered set, $(\mathbb{Q}, \leq)$ is the corrresponding reflexive totally ordered set.
- $(\mathbb{Q}, >)$ is a strict totally ordered set, where $>$ is $<^{-1}$. (That is, $x > y \;\leftrightarrow\; y < x$.)
- Let $\prec \subseteq \; \mathbb{Q} \times \mathbb{Q}$ be defined by

  $(x_1, y_1) \prec (x_2, y_2)$ iff $x_1 < x_2$ and $y_1 < y_2$.

  $\prec$ does **not** satisfy trichotomy on $\mathbb{Q}$; but does satisfy trichotomy on any line of positive slope (for example, $\{(x, 2x) \mid x \in \mathbb{Q}\}$.)

Example

- Let $|\subseteq \mathbb{N}\times\mathbb{N}$ be the divisibility relation defined by

$$m|n \quad \leftrightarrow \quad \exists q\in\mathbb{N}\,(n = qm)$$

- $(\mathbb{N}, |)$ is an ordered set (satisfying reflexivity), but not totally ordered: $2 \nmid 3 \wedge 3 \nmid 2 \wedge 2 \neq 3$.

Isomorphism

An isomorphism is a “structure-preserving map" between two ordered sets: a bijection which preserves the underlying ordering of both sets.

Definition

Let $(A,<)$ and (B,$\prec$) be ordered sets.

- F is an isomorphism from $(A,<)$ onto $(B,\prec)$ iff $F : A \rightleftarrows B$ and $\forall x,y\in A\big(x<y \leftrightarrow F(x)\prec F(y)\big)$.
- We will say that $(A,<)$ and $(B,\prec)$ are isomorphic, and write $(A,<)\cong(B,\prec)$ when there exists an isomorphism from $(A,<)$ onto $(B,\prec)$.

Note. Compare these order-preserving isomorphisms to group isomorphisms (bijections which preserve the group operator) and homeomorphisms (bijections which preserves the topology of open sets.

Examples of Isomorphisms

Sometimes the underlying order is left understood from the context:

- The map $F(n) = 2n$ is an isomorphism from $\mathbb{N}$ onto the even numbers.
- The map $F(x) = e^x$ is an isomorphism from $\mathbb{R}$ onto the positive real numbers, $(0, \infty)$.
- The map $F(n) = -n$ is an isomorphism from $(\mathbb{Z}, <)$ onto $(\mathbb{Z}, >)$.

Isomorphisms on total orders

Lemma

Let $(A, <)$ and $(B, \prec)$ be totally ordered sets, and $F : A \rightleftarrows B$ which additionally satisfies: $\forall x, y \in A\big(x < y \to F(x) \prec F(y)\big)$.

Then F is an isomorphism from $(A, <)$ onto $(B, \prec)$.

Proof.

We must show $\forall x, y \in A\big(F(x) \prec F(y) \to x < y\big)$. Fix $x, y \in A$ and suppose $F(x) \prec F(y)$. Since $<$ is a total order on A, we must have one of $x < y$ or $y < x$ or $x = y$.

☛ If $x = y$ then $F(x) = F(y)$, so $F(x) \prec F(x)$ which contradicts irreflexivity of $\prec$.

☛ If $y < x$ then $F(y) \prec F(x)$; since $F(x) \prec F(y)$, we have by transitivity $F(y) \prec F(y)$ which contradicts irreflexivity of $\prec$.

✓ Therefore, $x < y$. □

Isomorphic structures

☞ Isomorphic ordered sets have the same order properties.

Lemma

If $(A,<) \cong (B,\prec)$ and $<$ totally orders A, then $\prec$ totally orders B.

Proof.

Let $F : A \rightleftarrows B$ witness the isomorphism. Fix $x \neq y \in B$. Since F is a surjection, there are $a_x, a_y \in A$ with $F(a_x) = x, F(a_y) = y$.

Suppose $a_x < a_y$ (as $<$ totally orders A). Then,
$x = F(a_x) \prec F(a_y) = y$.
If $a_y < a_x$, then similarly, $y \prec x$.

So, $x \prec y$ or $y \prec x$. Therefore, $\prec$ totally orders B. □

Greatest and least elements

There are several notions of "greatest" and "least" elements on ordered sets.

Let $(A,<)$ be an ordered set and $B \subseteq A$.

- $b \in B$ is the least element of B if $b \leq x$ for every $x \in B$.
- $b \in B$ is a minimal element of B if $\neg\exists z \in B\,(z < b)$.
- $b \in B$ is the greatest element of B if $x \leq b$ for every $x \in B$.
- $b \in B$ is a maximal element of B if $\neg\exists z \in B\,(b < z)$.

Note. $b \in B$ is the greatest (a maximal) element of B in $(A,<)$ if and only if b is the least (resp. a minimal) element of B in $(A,>)$.

Convention. I will say an element is R-minimal (R-maximal) when I want to explicitly mention the underlying order R. The definitions of minimal and maximal make sense for an arbitrary relation R. (The main use will be when we study the Axiom of Foundation, where the relation R will be taken to be the membership relation, $\in$.)

Examples of least/minimal elements in orders

- 0 is the least element of $(\mathbb{N}, <)$; there is no greatest or maximal elements.
- The set $(0, 1)$ has no least or minimal elements (or greatest or maximal elements) in $(\mathbb{R}, <)$. The set $[0, 1]$ has the least element 0 and greatest element 1.
- 1 is the least element of $\mathbb{N}$ in $(\mathbb{N}, |)$; there is no greatest element.
- $\{2, 3, 4, 5, \ldots\}$ has no least element in $(\mathbb{N}, |)$; there are infinitely many minimal elements – every prime number is a minimal element.

Well-ordered sets

Definition

A relation R is well-founded on a set A iff for every non-empty $X \subseteq A$ there is a $y \in X$ which is R-minimal in X ($\neg \exists z \in X\, zRy$.)

Definition

An ordered set $(A, <)$ is a well-ordered set if $<$ is a total order and well-founded on A.

- The natural numbers, $(\mathbb{N}, <)$, is a well-ordered set. (We will prove this in Section 3.2.)
- The rational numbers, $(\mathbb{Q}, <)$ is not well-ordered: consider the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Subset of a well-founded set

Lemma

If R is well-founded on A and $B \subseteq A$, then $Q = R \cap (B \times B)$ is well-founded on B.

Proof.

Let $X \subseteq B$ and $X \neq \emptyset$. Then $X \subseteq A$, so let $a \in X$ be R-minimal. But, a is Q minimal as well, since $Q \subseteq R$. □

Isomorphic structures

☞ Another example of isomorphic ordered sets having the same order properties.

Lemma

If $(A, <) \cong (B, \prec)$ and $<$ well-orders A, then $\prec$ well orders B.

Proof.

Let $F : A \rightleftarrows B$ witness the isomorphism. Let $Y \subseteq B$ be nonempty, and $X = F^{-1}[Y]$. So, X is nonempty and has a $<$-least element a. We show that $F(a) = b$ is $\prec$-least in Y

Clearly, $b \in Y$. Suppose $c \in Y$. So, $F^{-1}(c) \in X$ and $a \leq F^{-1}(c)$, by the minimality of a in X. Thus, $b = F(a) \preceq c$.

Therefore, $\prec$ well-orders B. □

Well-ordered sets and bijections

☞ A set A is well-orderable if there exists an ordering $<$ on A such that $(A, <)$ is a well-ordered set.

Lemma

If A is well-orderable and there is a bijection $F : A \rightleftarrows B$, then B is well-orderable.

Proof.

Let $(A, <)$ be a well ordering. Define $\prec \subseteq B \times B$ by

$$x \prec y \ \leftrightarrow \ F^{-1}(x) < F^{-1}(y).$$

Now, F is an isomorphism from $(A, <)$ to $(B, \prec)$. So, $(B, \prec)$ is a well-ordering. □

Well-ordered sets and bijections

Example. Although $(\mathbb{Q}, <)$ is not a well-ordering, since $\mathbb{Q} \approx \mathbb{N}$ and $\mathbb{N}$ is well-orderable, there is an ordering $\prec$ of $\mathbb{Q}$ which is a well-ordering of $\mathbb{Q}$.

Example. $(\mathbb{R}, <)$ is not well-ordered. Is $\mathbb{R}$ well-orderable? Is there a well-ordered set S such that $S \approx \mathbb{R}$?

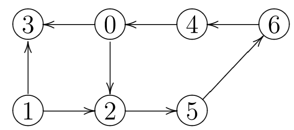
Can any set X be well-ordered?

☞ The statement that any set can be well-ordered turns-out to be equivalent to the Axiom of Choice, our Axiom 9. Without this axiom, we cannot prove that even $\mathbb{R}$ is well-orderable.

Zermelo's “construction" of a well-ordering of $\mathbb{R}$ drew alot of heat in 1904, since it depended on the Axiom of Choice.

Example of a well-founded set

Example. Consider the directed graph (where xRy if there is an arrow from x to y):



- $A = \{0, 1, 2, 3, 4\}$ is well-founded ($0, 1$ are R-minimal, $2, 3$ are R-maximal.)
- $A = \{0, 1, 2, 3, 4, 5\}$ is well-founded.
- $A = \{0, 1, 2, 3, 4, 5, 6\}$ is not well-founded because the set $X = \{0, 2, 5, 6, 4\}$ has no R-minimal element.
  The set X is called a cycle.

Note. In general, a relation R on a *finite* set A is well-founded iff R is acyclic (i.e. there are no R-cycles.)