Math 582
Introduction to Set Theory

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Relation to text

The material in this lecture covers Sections 2.1 (Ordered pairs), Section 2.2 (Relations) and section 2.3 (Functions) from Hrbacek and Jech.

Note. The text assumes the Power Set Axiom in developing the material in chapter two. We will develop this material using the Replacement Axiom instead.

We will not be able to justify Definition 3.13 (p. 26, and the discussion on p. 27), but will come back to this later when we need it.

Definitions

☞ Our "official" definition of ordered pair.
Definition. The ordered pair of sets x and y is

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$$

☞ The key property about ordered pair is that x, y are uniquely determined:

Theorem (Fundamental property of ordered pair)

For sets x, y, x', y'

$$\langle x, y\rangle = \langle x', y'\rangle \;\rightarrow\; x = x' \wedge y = y'$$

Proof of theorem

Proof.

Suppose $\langle x, y\rangle = \langle x', y'\rangle$.
Case $x = y$:

$$\langle x, x\rangle = \{\{x\}, \{x, x\}\} = \{\{x\}\} \text{ and so, } \{\{x\}\} = \{\{x'\}, \{x', y'\}\}$$

So, $\{x'\} = \{x\} = \{x', y'\}$ and thus $x' = y'$ and $x' \in \{x\}$, so $x' = x$ and $x = y'$.
Case $x \neq y$:

$$\{x\} = \{x'\} \text{ and } \{x, y\} = \{x', y'\}$$

So, $x = x'$ (first equality.) Since $y \in \{x', y'\}$ (second equality) and $y \neq x = x'$, we have $y = y'$. □

A word on ordered pairs

☞ Any particular definition of ordered pair which satisfies the Fundamental Property will work just as well. Such as (see Homework 5):

$$\langle\!\langle x, y \rangle\!\rangle = \{\{\emptyset, x\}, \{\{\emptyset\}, y\}\}$$

☞ It almost never matters what definition is used, provided that x and y are uniquely determined from the ordered pair. We will use the notation (x, y) for any set definition of ordered pair which satisfies the Fundamental Property.

☞ When we need to appeal to the specific definition of ordered pair, then we will revert back to our "official definition" of (x, y) as $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$.

Relations – defined

Definition

A set R is a (binary) relation iff R is a set of ordered pairs:

$$\forall u \in R \, \exists x, y \left[u = (x, y) \right]$$

Let R be a relation

- xRy abbreviates $(x, y) \in R$.
- $x \not R y$ abbreviates $(x, y) \notin R$.

Domain and Range of a relation

Definition

For any set R, define the sets

$$\begin{aligned} \mathrm{dom}(R) &= \{x \mid \exists y[(x,y) \in R]\} \\ \mathrm{ran}(R) &= \{y \mid \exists x[(x,y) \in R]\} \end{aligned}$$

Justification. From our definition of pair $(x,y) = \langle x,y\rangle$: if $\langle x,y\rangle \in R$ then

- $\{x\}, \{x,y\} \in \bigcup R$
- $x, y \in \bigcup\bigcup R$

By Comprehension

$$\begin{aligned} \mathrm{dom}(R) &= \{x \in \bigcup\bigcup R \mid \exists y[(x,y) \in R]\} \\ \mathrm{ran}(R) &= \{y \in \bigcup\bigcup R \mid \exists y[(x,y) \in R]\} \end{aligned}$$

Note. We can prove the existence of $\mathrm{dom}(R), \mathrm{ran}(R)$ which does not depend upon the specific definition of ordered pair. (See Homework 5.)

Restriction and image of relation

The following definitions are most frequently used with functions:

- The restriction of a relation R to a set A defined by

$$R \restriction A = \{(x,y) \in R \mid x \in A\}.$$

- The image of a set A under relation R defined by

$$R[A] = \{y \in \mathrm{ran}(R) \mid \exists x \in A\ (x,y) \in R\}.$$

Definition of function

Definition

A relation R is a function iff for every $x \in \mathrm{dom}(R)$ there is a unique y with $(x, y) \in R$.

We write $R(x)$ to denote this unique y.

Notation. I will usually use f, g, h, F, G, H for functions.

Basic definitions

- $F : A \to B$ means F is a function, $\mathrm{dom}(F) = A$, $\mathrm{ran}(F) \subseteq B$.
- $F : A \twoheadrightarrow B$ means $F : A \to B$ and $\mathrm{ran}(F) = B$. (We say that F is a surjection or F is onto.)
- $F : A \hookrightarrow B$ means $F : A \to B$ and $\forall x, x' \in A \left[f(x) = f(x') \;\rightarrow\; x = x'\right]$. (We say that F is an injection or F is one-to-one.)
- $F : A \rightleftarrows B$ means both $F : A \hookrightarrow B$ and $F : A \twoheadrightarrow B$. (We say that F is a bijection.)

Basic definitions

The following are true statements about the sine function on $\mathbb{R}$:

- $\sin : \mathbb{R} \to \mathbb{R}$
- $\sin[\mathbb{R}] = [-1, 1]$, $\sin[(0, \frac{\pi}{2})] = (0, 1)$, $\sin[(-\frac{\pi}{2}, 0)] = (-1, 0)$
- $\sin : \mathbb{R} \to [-1, 1]$
- $\sin : \mathbb{R} \twoheadrightarrow [-1, 1]$
- $\sin \restriction [-\frac{\pi}{2}, \frac{\pi}{2}] : [-\frac{\pi}{2}, \frac{\pi}{2}] \hookrightarrow \mathbb{R}$
- $\sin \restriction [-\frac{\pi}{2}, \frac{\pi}{2}] : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightleftarrows [-1, 1]$

Binary functions

☞ We defined functions on “one-argument" like sin from $\mathbb{R} \to \mathbb{R}$. We can define two-argument (binary) functions, such as $+$ on $\mathbb{N}$, using the cartesian products:

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \quad \text{by } + \big((n, m)\big) = n + m$$

where

$$\mathbb{N} \times \mathbb{N} = \{(n, m) \,\big|\, n, m \in \mathbb{N}\}$$

☞ We still need to define $\mathbb{N}$ (later) and $\mathbb{N} \times \mathbb{N}$ (to which we turn to now.)

Cartesian Product

☞ Given sets S, T we "define" the cartesian product

$$S \times T = \{(s,t) \mid s \in S \wedge t \in T\}.$$

This gives us a wealth of examples of relations and functions. Before we can accept this as a definition we must show that $S \times T$ exists for each S and T and the object is uniquely determined.

☞ At this point, we cannot even prove $\{0\} \times T$ exists (although *intuitively* this set "should exist", since it has the "same size" as T itself, so is not too big.)

☞ We need another axiom to construct sets like $S \times T$.

Replacement

Axiom 6. Replacement Scheme For each formula φ, without B free,

$$\forall x \in A\, \exists! y\, \varphi(x,y) \quad \to \quad \exists B \forall x \in A\, \exists y \in B\, \varphi(x,y)$$

Suppose for each $x \in A$ there is a unique y with $\varphi(x,y)$. Call this set y_x. The Replacement Scheme allows us to collect the various y_x to form the set $C = \{y_x \mid x \in A\}$.

☞ Replacement and Comprehension allows us to construct this set C:

$$C = \{y \in B \mid \exists x \in A\, \varphi(x,y)\}$$

Cartesian Product

Definition. Given sets S, T

$$S \times T = \{(s,t) \mid s \in S \wedge t \in T\} = \{u \mid \exists s \in S \exists t \in T\, u = (s,t)\}$$

Justification. Use Replacement and Comprehension

1. Fix $s \in S$ and form $\{s\} \times T$ by

$$\{s\} \times T = \{u \mid \exists t \in T\, u = (s,t)\}$$

   (For each $t \in T$ there exists a unique set u with $u = (s,t)$.)
2. Let $D = \{\{s\} \times T \mid s \in S\}$ using Replacement and Comprehension. (Note that for each $s \in S$ there is a unique set $\{s\} \times T$.)
3. Now let

$$S \times T = \bigcup D = \bigcup_{s \in S} \{s\} \times T.$$

Defining functions with Replacement

A common use of Replacement is to define functions:

Lemma

Suppose $\forall x \in A \exists! y\, \varphi(x,y)$. Then there is a function F with $dom(F) = A$ and for each $a \in A$, $\varphi(a, F(a))$.

Proof.

Fix B as in Replacement, and let

$$F = \{(x,y) \in A \times B \mid \varphi(x,y)\}.$$

□

Ternary relations and binary functions

☞ We can define ternary relations as sets on ordered triples. We write

$$A \times B \times C := (A \times B) \times C$$

and (a, b, c) for $((a, b), c)$.

☞ Two-argument functions are simply ternary relations which satisfy the condition of functionality:

$F : A \times B \to C$ means $F \subseteq A \times B \times C$ and satisfies $\forall a \in A, b \in B \, \exists! c \in C \, \big[(a, b, c) \in F\big]$. So, $\text{dom}(F) = A \times B$ and $\text{ran}(F) \subseteq C$

Example. $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

Examples

- Although $\in \subseteq V \times V$ is not “formally" a relation, it is if we consider $\in \subseteq A \times A$ for a set A:

$$\in_A = \{(a, b) \in A \times A \,\big|\, a \in b\}$$

- Although $\cap : V \times V \to V$ is not “formally" a function, it is if we consider $\cap \restriction A \times A$ for a set A:

$$\cap_A = \{((a, b), c) \,\big|\, (a, b) \in A \times A \wedge a \cap b = c\}$$

Inverse and composition

- The inverse of a relation R is defined as

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

Justification. $R^{-1} \subseteq \text{ran}(R) \times \text{dom}(R)$, now apply Comprehension.

- The composition of relations R, S is defined by

$$S \circ R = \{(x, z) \mid \exists y\big[(x, y) \in R \wedge (y, z) \in S\big]\}$$

Justification. $S \circ R \subseteq \text{dom}(R) \times \text{ran}(S)$.

Examples of Inverse

☞ Consider the relation $<$ on $\mathbb{N}$. Then, $<^{-1} = >$ on $\mathbb{N}$.

$$27 <^{-1} 5 \text{ since } 27 > 5$$

☞ Define $R \subseteq \mathbb{N} \times \mathbb{N}$ by

$$R = \{(m, n) \mid \exists k \; m \cdot k = n\}.$$

That is, mRn iff m divides n.

Then, the inverse relation is

$$R^{-1} = \{(n, m) \mid \exists n = k \; m \cdot k\}.$$

That is, $nR^{-1}m$ iff n is a multiple of m.

Examples of Compositions

☞ Consider again $\leq$ on $\mathbb{N}$. Then

$$(\leq \circ \leq) = \leq,$$

since $\leq$ is reflexive: $n \leq n$, and transitive:

$$k \leq m \wedge m \leq n \rightarrow k \leq n \quad \text{for all } k, m, n.$$

☞ On the other hand, for $<$ on $\mathbb{N}$,

$$(< \circ <) \neq < .$$

For example $0 < 1$, but it is NOT true that $0(< \circ <)1$.

Basic Properties of Composition and Inverse

Lemma

Let $R \subseteq X \times Y$.

(a) $R[X] = ran(R)$, $\qquad R^{-1}[Y] = dom(R)$,

(b) $(R^{-1})^{-1} = R$,

(c) $dom(R) = ran(R^{-1})$, $\qquad ran(R) = dom(R^{-1})$,

(d) $R^{-1} \circ R \supseteq Id_{dom(R)}$, $\qquad R \circ R^{-1} \supseteq Id_{ran(R)}$.

where $Id_Z = \{(x, x) \mid x \in Z\}$.

Proofs

Let $R \subseteq X \times Y$.

(a).

$$\begin{aligned} y \in R[X] &\leftrightarrow \exists x \in X\ (x, y) \in R \\ &\leftrightarrow y \in \mathrm{ran}(R) \\ x \in R^{-1}[Y] &\leftrightarrow \exists y \in Y\ (y, x) \in R^{-1} \\ &\leftrightarrow \exists y \in Y\ (x, y) \in R \\ &\leftrightarrow x \in \mathrm{dom}(R). \end{aligned}$$

(b).

$$(x, y) \in R \leftrightarrow (y, x) \in R^{-1} \leftrightarrow (x, y) \in (R^{-1})^{-1}$$

(c). By (a)

$$\mathrm{ran}(R^{-1}) = R^{-1}[Y] = \mathrm{dom}(R),$$

and by (b)

$$\mathrm{ran}(R) = R[X] = (R^{-1})^{-1}[X] = \mathrm{dom}(R^{-1}),$$

Proofs

(d).

$$\begin{aligned} (x, x) \in \mathrm{Id}_{\mathrm{dom}(R)} &\leftrightarrow \exists y \in Y\ (x, y) \in R \\ &\leftrightarrow \exists y\ (x, y) \in R \wedge (y, x) \in R^{-1} \\ &\rightarrow (x, x) \in R^{-1} \circ R. \\ (y, y) \in \mathrm{Id}_{\mathrm{ran}(R)} &\leftrightarrow \exists x \in X\ (x, y) \in R \\ &\leftrightarrow \exists x \in X\ (x, y) \in R \wedge (y, x) \in R^{-1} \\ &\rightarrow (y, y) \in R \circ R^{-1}. \end{aligned}$$

Inverse of a function

Definition. A function F is invertible if and only if F^{-1} is a function.

Lemma

F is invertible if and only if it is one-to-one. Furthermore, if F is invertible, the F^{-1} is invertible and $(F^{-1})^{-1} = F$.

Proof

Proof.

☞ Suppose F is invertible. Then F^{-1} is a function, so for each $x \in \text{dom}(F)$, $F^{-1} \circ F(x) = x$. Then

$$F(x) = F(y) \;\rightarrow\; F^{-1} \circ F(x) = F^{-1} \circ F(y) \;\rightarrow\; x = y.$$

☞ Suppose F is injective. Then

$$\begin{aligned} yF^{-1}x \,\wedge\, yF^{-1}z \quad &\rightarrow \quad xFy \,\wedge\, zFy \\ &\rightarrow \quad F(x) = y \,\wedge\, F(z) = y \\ &\rightarrow \quad x = z. \end{aligned}$$

☞ $F = (F^{-1})^{-1}$ follows from the previous Lemma.

□

Composition of functions

Lemma

Let F and G be functions. Then $G \circ F$ is also a function.

When F, G are functions and $ran(F) \subseteq dom(G)$ we write $(G \circ F)(x)$ as $G(F(x))$.

Proof.

$$
\begin{aligned}
x(G \circ F)y \ \wedge \ x(G \circ F)z &\rightarrow \exists u, v \ xFu \ \wedge \ uGy \ \wedge \ xFv \ \wedge \ vGz \\
&\rightarrow u = v \qquad F \text{ is a function} \\
&\rightarrow y = z \qquad G \text{ is a function}
\end{aligned}
$$

□