# Math 582
# Introduction to Set Theory

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## Axiomatic Set Theory

☞ Informally, our universe of sets is the class of all hereditary sets:

- If $x$ is a set, then all members of $x$ are sets, and all members of members of $x$ are sets, and so forth.

☞ Formally, we are just exhibiting a list of sentences (infinitely many) in first-order predicate logic, with one relation symbol, $\in$.

☞ It is possible to ignore the “intended interpretation" and take the axioms as just making assertions about a binary relation on a nonempty domain.

## Interpretation Set Theory

☞ It is useful to think of finite interpretations of set and membership as a directed graph (digraph) where

- Node are sets, and
- The edges determine membership: the children of a set are the members of the set.

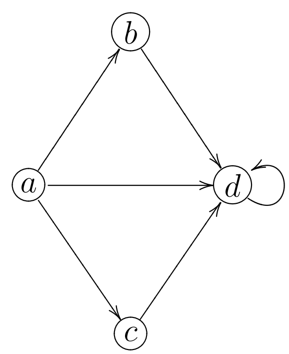
☞ We can think determine use these toy interpretations to show the independence of the axioms from each other. If we can show there is an interpretation where axiom $A$ is false and axioms $B, C, \ldots D$ are true, then we know we cannot derive $A$ from $B, C, \ldots, D$.

☞ See Homework set 4 for more examples of this.





## Example 1



## Example 1

☞ The following axioms are true in this digraph

- Foundation (2): It may look like $d = \{a, b, c, d\}$ is a counterexample, but $a \in d$ and $a \cap d = \emptyset$.
- Pairing (4), Union (5): The presence of a "universal set" $d$ is sufficient to guarantee these hold.

☞ The following axioms are false in this digraph

- Extensionality (1): $b = \{a\} = c$, but $b \neq c$.
- Comprehension (3): The existence of a universal set is sufficient here (which must fail in a model where Comprehension holds). Another example of the failure is that $\{b\}$ is definable by Comprehension, but there is no such set in the interpretation.





## Example 2

☞ An infinite model consists of $(\mathbb{N}, <)$, the natural numbers with $<$ interpreting membership. For example, $0 \in 1$, $5 \in 27$, etc. Also, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, etc.

☞ The following axioms are true:

- Extensionality (1), Foundation (2)
- Pairing (4): For any numbers $k, \ell$, let $m = \max(k, \ell) + 1$, so $k, \ell \in m$.
- Union (5): $\bigcup m = m - 1$ (or 0 if $m \leq 1$).

☞ The following axioms are false:

- Comprehension (3): There are (in general) no ordered pairs, such as $\{1, 2\}$ (and 3 is big enough to satisfy the pairing axiom).

## Example 3

☞ Another infinite model consists of $(\mathbb{Z}, <)$, the integers with $<$ interpreting membership. Now all sets are infinite. For example

$$0 = \{\ldots, -3, -2, -1\}.$$

☞ The following axioms are true:

- Extensionality (1)
- Pairing (4): For any numbers $k, \ell$, let $m = \max(k, \ell) + 1$, so $k, \ell \in m$.
- Union (5): $\bigcup m = m - 1$.

☞ The following axioms are false:

- Comprehension (3): There are (in general) no ordered pairs, such as $\{1, 2\}$ (and 3 is big enough to satisfy the pairing axiom).
- Foundation (2): $0 \neq \emptyset$ and for any $n < 0$

$$n \cap 0 = \{n-1, n-2, \ldots\} \neq \emptyset.$$





## Example 4

☞ A recursive definition of hereditarily finite set:

- $\emptyset$ is a hereditarily finite set.
- If $a_1, \ldots, a_k$ are hereditarily finite sets, then so is $\{a_1, \ldots, a_k\}$.

Or equivalently,

$$\begin{aligned} V_0 &= \emptyset \\ V_{n+1} &= \mathcal{P}(V_n) \\ V^* &= \bigcup_{n=0}^{\infty} V_n. \end{aligned}$$

☞ All axioms of ZFC, except the Axiom of Infinity (Axiom 7) are true in this model.