Math 582
Introduction to Set Theory

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Course Data

- **Text**. *Introduction to Set Theory* by Karel Hrbacek and Thomas Jech (3rd. ed.)
  We will cover Chapters 1-9.
- **Office**. 1842 East Hall
- **Office hours**. 10-11, 1-2 (M,W,F) and by appointment
- **Web**. http://www-personal.umich.edu/~kaharri/582/
  Linked off CTools
- **Homework**. Assignments due every two weeks.

Naive Set Theory

Naive set theory supports the everyday usage of set concepts in most branches of contemporary mathematics.

Properties.

- Nonformal. It uses the natural language and notation of ordinary informal (or semiformal) mathematics.
- Intuitive. The set concepts and their basic properties are assumed to be understood.
- Restricted in scope. Naive set theory supports a particular branch of mathematics – such as analysis, topology, algebra.

Axiomatic Set Theory

Axiomatic set theory is an autonomous mathematical discipline dedicated to the study of the universe of sets.

Properties.

- Formal. The language of axiomatic set theory is formalized in first-order logic. (A formal language is a mathematical object, and is the object of study by by Logicians.)
- Axiomatic. Set and set membership are treated as undefined terms, whose properties are given solely through a collection of axioms. These axioms can be quite unintuitive (or even counterintuitive!)
- Universal in scope. Set theory is a foundation for all of ordinary mathematics, including itself. Every mathematical object is a set.

Naive vs Axiomatic point of view

☞ Two points of view about the primitive terms of Geometry (points, straight lines, and planes).

- Euclid through nineteenth century (intuitive view).
  - A point is that which has no part.
  - A straight line is a line which lies evenly with the points on itself.
  - A plane a surface which lies evenly with the straight lines on itself.

  Geometry depends upon intuitive knowledge of Euclidean space.
- Twentieth century (axiomatic view).

> *One must always be able to say instead of 'points, straight lines and planes', 'tables, chairs, and beer mugs'.*

(From David Hilbert's *Grundlagen der Geometrie*, 1891)

Logical notation

We will use the following standard logical symbols to abbreviate English expressions:

- $\vee$ abbreviates "or",
- $\wedge$ abbreviates "and"
- $\neg$ abbreviates "not"
- $\rightarrow$ abbreviates "implies" ("if-then")
- $\leftrightarrow$ abbreviates "iff" ("if and only if")
- $\forall x$ abbreviates "for every object x",
- $\exists x$ abbreviates "there exists an object x".

Naive concept of set

☞ The set concept is fundamental and not reducible to simpler concepts.

☞ Georg Cantor described it as follows:

> *By a set we are to understand any collection into a whole of definite and separate objects of our intuition or our thought.*

Some basic properties of sets

Cantor's description implies three basic properties of sets.

❶ Every set A has elements or members. We write

$$x \in A \leftrightarrow \text{ the object } x \text{ is a member of } A.$$

❷ A set is determined by its members. If A and B are sets, then

$$A = B \leftrightarrow \forall x \big[x \in A \leftrightarrow x \in B \big].$$

This is called the Extensionality Property.

❸ A set is a collection of objects sharing a common property.
For every definite property **P** there is a set A such that

$$\forall x \big[x \in A \leftrightarrow \mathbf{P} \text{ is true of } x \big] \qquad \text{abbrev. as } \mathbf{P}(x).$$

This is called the Naive Comprehension Property.

Example: Empty set

Example. Let **P** be the property $x \neq x$
(that is, "x is not identical to itself", a property true of no objects).

☞ By Comprehension, there is a set $\emptyset$ with no members:

$$\forall x\big[x \in \emptyset \;\leftrightarrow\; x \neq x\big]$$

☞ By Extensionality, there is only one emptyset.
If A is any other set with no members, then $A = \emptyset$.

Definition. $\emptyset$ is the unique set with no members.

Comprehension and uniqueness

Theorem. Let $\mathbf{P}(x)$ be a definition property. Then there is a unique set A satisfying

$$\forall x\big[x \in A \;\leftrightarrow\; \mathbf{P}(x)\big].$$

Proof. There exists a set A by Comprehension. Suppose B is given by

$$\forall x\big[x \in B \;\leftrightarrow\; \mathbf{P}(x)\big].$$

Then $A = B$ by Extensionality: fix any object x, then

$$x \in A \;\leftrightarrow\; \mathbf{P}(x) \;\leftrightarrow\; x \in B.$$

so, $\forall x\big[x \in A \;\leftrightarrow\; x \in B\big]$.

Set Notation

Notation. If the number of objects is finite, we specify the set by listing its members. We write

$$A = \{a_1, \dots, a_n\}$$

when A contains the objects $a_1, \dots, a_n$ and nothing else.

☞ This set exists by Comprehension:

let $\mathcal{P}(x)$ be $x = a_1 \vee \dots \vee x = a_n$.

Examples.

- Suits: $\{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\}$,
- Singleton of a: $\{a\}$,
- Unordered pair of a, b: $\{a, b\}$

finite lists

☞ Extensionality implies that once we specify a set by listing its objects, order and repetitions do not matter.

Examples.

- The following sets are identical.

$$\{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\} \qquad \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$$

- The following sets are identical.

$$\{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\} \qquad \{\clubsuit, \clubsuit, \heartsuit, \diamondsuit, \spadesuit, \spadesuit\}$$

Notation

Notation. A set can be specified by stating a definite property of its objects. We write

$$A = \{x \mid \mathbf{P}(x)\}$$

when A is the set of all objects x for which $\mathbf{P}(x)$ holds.

☞ This set exists by Comprehension.

Example. The following sets are identical

$$\{\heartsuit, \diamondsuit\} \qquad \{x \mid x \text{ is a red suit }\}.$$

Infinite sets

☞ This notation is especially useful for infinite sets.

Examples.

$$\begin{aligned}
\mathbb{N} &= \{x \mid x \text{ is a natural number }\} \\
\mathbb{Z} &= \{z \mid z \text{ is an integer }\} \\
\mathbb{Q} &= \{\frac{a}{b} \mid a, b \in \mathbb{Z} \ \wedge \ b \neq 0\} \\
\mathbb{R} &= \{r \mid r \text{ is a real number }\} \\
\mathbb{C} &= \{a + bi \mid a, b \in \mathbb{R} \ \wedge \ i = \sqrt{-1}\}
\end{aligned}$$

Infinite lists

Notation. We sometimes specify a set by listing some of its objects, and let the context determine the pattern. We write

$$A = \{a_1, a_2, a_3 \ldots\}$$

when A contains precisely a_1, a_2, a_3 as well as other objects determined by the context.

Example. This notation is often used for large finite or infinite sets.

- $\{0, 1, 2, 3, \ldots\} = \mathbb{N}$,
- $\{0, 2, 4, 6, \ldots\} = \{x \mid x \in \mathbb{N}$ and x is even $\}$,
- $\{2, 3, 5, 7, 11, 13, \ldots\} = \{x \mid x$ is one of a twin prime $\}$.

Definition of subset

Definition. Let A and B be sets. We say A is a subset of B, and write $A \subseteq B$, when every element of A is an element of B:

$$A \subseteq B \leftrightarrow \forall x \big[x \in A \rightarrow x \in B\big].$$

We write $A \subset B$ when A is a subset of B, but not equal to B:

$$A \subset B \leftrightarrow \big[A \subseteq B \wedge A \neq B\big].$$

Examples.

$$\{\heartsuit, \diamondsuit\} \subseteq \{\clubsuit, \heartsuit, \diamondsuit, \spadesuit\}$$

$$\{0, 2, 4, \ldots\} \subseteq \mathbb{N}$$

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

We can replace $\subseteq$ with $\subset$ in these examples.

Simple properties of subset

Proposition. For all sets A, B and C, the following properties hold

Reflexivity	$A \subseteq A$
Symmetry	$[A \subseteq B \wedge B \subseteq A] \rightarrow A = B$
Transitivity	$[A \subseteq B \wedge B \subseteq C] \rightarrow A \subseteq C$

Note. Any relation (such as $\subseteq$) which satisfies these three properties is said to be an equivalence relation.

Example. $\leq$ on $\mathbb{N}$.

Smallest set

☞ There is a smallest set (relative to ordering by $\subseteq$).

Proposition. $\emptyset \subseteq A$ for every set A.

Question. Is there a largest set V:

$$A \subseteq V \qquad \text{for every set } A?$$

- A set V with this property is called a universal set.
- Would a universal set be unique?