# MATH 582 HOMEWORK 5
# WEEK 9

*Winter, 2009*
*Due March 27*

**Exercise 1.** *Prove the distributive law : for all* $\alpha, \beta, \gamma$,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

**Exercise 2.** *Prove that ordinal exponentiation satisfies*
*(a)* $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$.
*(b)* $\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{(\beta \cdot \gamma)}$.

**Exercise 3.** *For every ordinal* $\alpha$*, there is a unique limit ordinal* $\gamma$ *(or* $\gamma = 0$*, if* $\alpha < \omega$*) and a natural number* $n$ *such that* $\alpha = \beta + n$*.*

**Exercise 4.** *Prove the following are equivalent.*
*(i)* $\alpha$ *is a limit ordinal.*
*(ii)* $\alpha = \omega \cdot \beta$ *for some* $\beta > 0$*.*
*(iii) For every nonzero* $m \in \omega$, $m \cdot \alpha = \alpha$ *and* $\alpha \neq 0$*.*

**Exercise 5.** *If* $(A, R)$ *is a well-ordered set and* $X \subseteq A$ *then type*$(X, R) \leq$ *type*$(A, R)$*. (Hint. We have already shown that* $(X, R)$ *is a well-ordered set. You may assume* $X \subseteq A \in \boldsymbol{ON}$ *and* $R$ *is* $<$ *(Why?) Consider an isomorphism* $f : X \to \delta$ *where type*$(X, <) = \delta$ *and show that* $f(\xi) \leq \xi$ *by transfinite induction on* $\xi$*.)*

**Exercise 6.** *Show that* $\alpha \cdot \beta =$ *type*$(\beta \times \alpha)$ *for all* $\alpha, \beta$*.*

**Exercise 7.** *Let* $\alpha$ *be a limit ordinal. Show the following are equivalent (such* $\alpha$ *are called indecomposable ordinals):*
*(a)* $\forall \beta < \alpha \left(\beta + \alpha = \alpha\right)$.
*(b)* $\forall \beta, \gamma < \alpha \left(\beta + \gamma < \alpha\right)$.
*(c)* $\exists \delta \left(\alpha = \omega^{\delta}\right)$.
*(d)* $\forall X \subseteq \alpha \left(\text{type}(X) = \alpha \ \vee \ \text{type}(\alpha - X) = \alpha\right)$.

*Hint.* (b) $\Rightarrow$ (c): *Use the bracketing theorem (Lecture 20, slide 13) with the function* $\left(\delta \mapsto \omega^\delta\right)$. *For* (c) $\Rightarrow$ (d). *Prove by transfinite induction on* $\delta$. *Consider* $X$ *and* $X^c$ *where* $X^c = \omega^\delta - X$. *For the successor case, where* $\omega^\delta = \omega^{\gamma+1}$, *break* $\omega^\delta$ *into a disjoint sequence of intervals of length* $\omega^\gamma$ *and consider the types of* $X$ *and* $X^c$ *restricted on these intervals. For the limit case,* $\omega^\delta = \sup\{\omega^\xi \mid \xi < \delta\}$, *so take the disjoint intervals* $[\omega^\xi, \omega^{\xi+1})$ *and consider the types of* $X$ *and* $X^c$ *when restricted to each of these intervals. In each case the intervals are disjoint and unbounded in* $\omega^\delta$, *that is for each interval there is an interval above it. The inductive hypothesis applies to each of these intervals. Use these intervals to show that the type of* $X$ *or* $X^c$ *must be* $\omega^\delta$.

**Exercise 8.** *Prove that the following combinatorial definition of ordinal exponentiation is equivalent to the definition by transfinite recursion. Let*

$$A(\alpha,\beta) = \left\{ f \in {}^{\beta}\alpha \;\middle|\; \{\xi \mid f(\xi) \neq 0\} \text{ is finite } \right\}.$$

*For* $f, g \in A(\alpha,\beta)$ *and* $f \neq g$, *say* $f \lhd g$ *iff* $f(\xi) < g(\xi)$, *where* $\xi$ *is the largest ordinal such that* $f(\xi) \neq g(\xi)$. *Show that* $\alpha^\beta = \text{type}\big(A(\alpha,\beta), \lhd\big)$.