MATH 582 HOMEWORK 5
WEEK 9

Winter, 2009

Due March 27

Exercise 1. *Prove the distributive law : for all α, β, γ,*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

Proof. The proof is by transfinite induction on γ. When $\gamma = 0$ both sides of the equality are 0. Suppose $\gamma = \delta + 1$, and that (i.h.)

$$\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta.$$

Then

$$\begin{aligned}
\alpha \cdot (\beta + \gamma) &= \alpha \cdot (\beta + (\delta + 1)) \\
&= \alpha \cdot ((\beta + \delta) + 1)) \\
&= \alpha \cdot (\beta + \delta) + \alpha && \text{def. of mult.} \\
&= \alpha \cdot \beta + \alpha \cdot \delta + \alpha && \text{i.h.} \\
&= \alpha \cdot \beta + \alpha \cdot (\delta + 1) && \text{def. of mult.} \\
&= \alpha \cdot \beta + \alpha \cdot \gamma.
\end{aligned}$$

Suppose γ is a limit and for all $\delta < \gamma$ (i.h.)

$$\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta.$$

Consider the following normal functions (Lecture 20, slide 7)

$$\begin{aligned}
F_\alpha(\xi) &= \alpha \cdot \xi \\
G_{\alpha\cdot\beta}(\xi) &= \alpha \cdot \beta + \xi
\end{aligned}$$

So, $G_{\alpha\cdot\beta} \circ F_\alpha$ is also normal (Lecture 20, slide 11).

Note that $\beta + \gamma$ is a limit and that

$$\beta + \gamma = \sup\{\beta + \delta \,\big|\, \delta < \gamma\}.$$

Since F_α is normal

$$\alpha\cdot(\beta+\gamma) = F_\alpha(\beta+\gamma) = \sup\{F_\alpha(\beta+\delta) \,\big|\, \delta < \gamma\} = \sup\{\alpha\cdot(\beta+\delta) \,\big|\, \delta < \gamma\}$$

So,

$$\begin{aligned}
\alpha \cdot (\beta + \gamma) &= \sup\{\alpha \cdot (\beta + \delta) \,\big|\, \delta < \gamma\} \\
&= \sup\{\alpha \cdot \beta + \alpha \cdot \delta) \,\big|\, \delta < \gamma\} && \text{i.h.} \\
&= \sup\{G_{\alpha\cdot\beta} \circ F_\alpha(\delta) \,\big|\, \delta < \gamma\} \\
&= G_{\alpha\cdot\beta} \circ F_\alpha(\delta) && \text{continuity} \\
&= \alpha \cdot \beta + \alpha \cdot \gamma.
\end{aligned}$$

$\square$

Exercise 2. *Prove that ordinal exponentiation satisfies*
(a) $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$.
(b) $\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{(\beta\cdot\gamma)}$.

Proof. (a). The proof is by transfinite induction on γ. When $\gamma = 0$, both sides of the equality are α^{β}. Suppose $\gamma = \delta + 1$ and that (i.h.)

$$\alpha^{\beta+\delta} = \alpha^{\beta} \cdot \alpha^{\delta}.$$

Then

$$\begin{aligned}
\alpha^{\beta+\gamma} &= \alpha^{\beta+(\delta+1)} \\
&= \alpha^{(\beta+\delta)+1} \\
&= \alpha^{\beta+\delta} \cdot \alpha && \text{def. of exp.} \\
&= \alpha^{\beta} \cdot \alpha^{\delta} \cdot \alpha && \text{i.h.} \\
&= \alpha^{\beta} \cdot \alpha^{\delta+1} \\
&= \alpha^{\beta} \cdot \alpha^{\gamma}.
\end{aligned}$$

Suppose γ is a limit and for all $\delta < \gamma$ (i.h.)

$$\alpha \cdot (\beta + \delta) = \alpha \cdot \beta + \alpha \cdot \delta.$$

Consider the following normal functions (Lecture 20, slide 7)

$$\begin{aligned}
F_{\alpha}(\xi) &= \alpha^{\xi} \\
G_{\alpha\cdot\beta}(\xi) &= \alpha \cdot \beta \cdot \xi
\end{aligned}$$

So, $G_{\alpha\cdot\beta} \circ F_{\alpha}$ is also normal (Lecture 20, slide 11).

Note that $\beta + \gamma$ is a limit and that

$$\beta + \gamma = \sup\{\beta + \delta \,\big|\, \delta < \gamma\}.$$

Since F_{α} is normal

$$\alpha^{\beta+\gamma} = F_{\alpha}(\beta+\gamma) = \sup\{F_{\alpha}(\beta+\delta) \,\big|\, \delta < \gamma\} = \sup\{\alpha^{\beta+\delta} \,\big|\, \delta < \gamma\}$$

So,

$$\begin{aligned}
\alpha^{\beta+\gamma} &= \sup\{\alpha^{\beta+\delta} \,\big|\, \delta < \gamma\} \\
&= \sup\{\alpha^{\beta} \cdot \alpha^{\delta} \,\big|\, \delta < \gamma\} && \text{i.h.} \\
&= \sup\{G_{\alpha^{\beta}} \circ F_{\alpha}(\delta) \,\big|\, \delta < \gamma\} \\
&= G_{\alpha^{\beta}} \circ F_{\alpha}(\delta) && \text{continuity} \\
&= \alpha^{(\beta\cdot\gamma)}.
\end{aligned}$$

(b). The proof is by transfinite induction on γ. When $\gamma = 0$, both sides of the equality are 1. Suppose $\gamma = \delta + 1$ and that (i.h.)

$$\left(\alpha^\beta\right)^\delta = \alpha^{\beta\cdot\delta}.$$

Then

$$\begin{aligned}
\left(\alpha^\beta\right)^\gamma &= \left(\alpha^\beta\right)^{\delta+1} \\
&= \left(\alpha^\beta\right)^\delta \cdot \alpha^\beta \quad \text{def. of exp.} \\
&= \alpha^{\beta\cdot\delta} \cdot \alpha^\beta \quad \text{i.h.} \\
&= \alpha^{\beta\cdot\delta+\beta} \quad \text{(a)} \\
&= \alpha^{\beta\cdot(\delta+1)} \\
&= \alpha^{\beta\cdot\gamma}
\end{aligned}$$

Suppose γ is a limit and for all $\delta < \gamma$ (i.h.)

$$\left(\alpha^\beta\right)^\delta = \alpha^{\beta\cdot\delta}.$$

Consider the following normal functions (Lecture 20, slide 7)

$$\begin{aligned}
F_\beta(\xi) &= \beta^\xi \\
G_\alpha(\xi) &= \alpha^\xi
\end{aligned}$$

So, $G_\alpha \circ F_\beta$ is also normal (Lecture 20, slide 11).

Note that $\beta + \gamma$ is a limit and that

$$\beta + \gamma = \sup\{\beta + \delta \,|\, \delta < \gamma\}.$$

So,

$$\begin{aligned}
\left(\alpha^\beta\right)^\gamma &= \sup\{\left(\alpha^\beta\right)^\delta \,|\, \delta < \gamma\} \quad \text{def. of exp.} \\
&= \sup\{\alpha^{\beta\cdot\delta} \,|\, \delta < \gamma\} \quad \text{i.h.} \\
&= \sup\{G_\alpha \circ F_\beta(\delta) \,|\, \delta < \gamma\} \\
&= G_\alpha \circ F_\beta(\gamma) \quad \text{continuity} \\
&= \alpha^{\beta\cdot\gamma}
\end{aligned}$$

□

Exercise 3. *For every ordinal α, there is a unique limit ordinal γ (or $\gamma = 0$, if $\alpha < \omega$) and a natural number n such that $\alpha = \beta + n$.*

Proof. If $\alpha < \omega$, then $\alpha = n$ for some n, so $\alpha = 0 + n$.

Suppose $\alpha \geq \omega$. Since (by Lecture 20, slide 19) the function Λ enumerating the limit ordinals is normal and $\Lambda(0) \leq \alpha$, for each α, there exists a unique β with $\Lambda(\gamma) \leq \alpha < \Lambda(\gamma+1)$ (Bracketing Theorem, Lecture 20, slide 13).

By the successor case for Λ:

$$\Lambda(\beta) \leq \alpha < \Lambda(\gamma) + \omega.$$

Let $n \in \omega$ and $\beta = \Lambda(\gamma)$. Then $\alpha = \beta + n$.

Uniqueness of representation. If $\beta' < \beta$ then $\beta' + m < \beta$ for every m, since β is a limit; if $\beta < \beta'$ and β' is a limit, then $\beta + m < \beta'$ for every m. Thus, β is uniquely determined. That n is uniquely determined follows from the Order Property of addition. □

Exercise 4. *Prove the following are equivalent.*

(i) α *is a limit ordinal.*

(ii) $\alpha = \omega \cdot \beta$ *for some* $\beta > 0$.

(iii) For every nonzero $m \in \omega$, $m \cdot \alpha = \alpha$ *and* $\alpha \neq 0$.

Proof. $(i) \rightarrow (ii)$. Let α be a limit ordinal. Consider the function $F : \mathbf{ON} \rightarrow \mathbf{ON}$ defined by recursion:

$$F(0) = 0 \qquad F(\beta) = \omega \cdot \beta \quad \text{for } \beta > 0.$$

This function is continuous: let γ be a limit ordinal, then

$$\begin{aligned} F(\gamma) &= \omega \cdot \gamma \\ &= \sup\{\omega \cdot \xi \,|\, \xi < \gamma\} && \text{Def. of } \cdot \\ &= \sup\{F(\xi) \,|\, \xi < \gamma\} && \text{Def. of } F. \end{aligned}$$

The function is order-preserving (we need only check successors by Lecture 20, slide 9)

$$\begin{aligned} F(\beta) &= \omega \cdot \beta \\ &< \omega \cdot \beta + \omega \\ &= \omega \cdot (\beta + 1) \\ &= F(\beta + 1). \end{aligned}$$

So, F is normal.

By the Bracketing Theorem (Lecture 20, slide 13), there is a unique β such that

$$F(\beta) \leq \alpha < F(\beta + 1) \quad \text{equivalently} \quad \omega \cdot \beta \leq \alpha < \omega \cdot (\beta + 1)$$

We must have $\alpha = \omega \cdot \beta$: since

$$\omega \cdot (\beta + 1) = \omega \cdot \beta + \omega = \sup\{\omega \cdot \beta + n \,|\, n \in \omega\}$$

so, if $\omega \cdot \beta < \alpha < \omega \cdot (\beta + 1)$ there is an $0 < n \in \omega$ with $\alpha = \omega \cdot \beta + n$, which is a successor ordinal, contradicting our assumption α is a limit ordinal. Furthermore, $\beta > 0$, since $\omega \cdot 0 = 0$ and $\alpha > 0$.

$(ii) \to (iii)$. Let $\alpha = \omega \cdot \beta$ for some $\beta > 0$. Then
$$n \cdot \alpha = n \cdot (\omega \cdot \beta) = (n \cdot \omega) \cdot \beta = \omega \cdot \beta,$$
Since $n \cdot \omega = \omega$ for every $n > 0$.

$(iii) \to (i)$. Suppose (iii) holds but (i) fails. Then, $\alpha = S(\beta)$ for some β. Then
$$2 \cdot \alpha = 2 \cdot (\beta + 1) = 2 \cdot \beta + 2 \geq \beta + 2 > \beta + 1 = \alpha.$$
So, $2 \cdot \alpha \neq \alpha$. This contradicts (iii). So, (i) holds.

□

Exercise 5. *If (A, R) is a well-ordered set and $X \subseteq A$ then type$(X, R) \leq$ type(A, R). (Hint. We have already shown that (X, R) is a well-ordered set. You may assume $X \subseteq A \in$ **ON** and R is $<$ (Why?) Consider an isomorphism $f : X \to \delta$ where type$(X, <) = \delta$ and show that $f(\xi) \leq \xi$ by transfinite induction on ξ.)*

Proof. Let type$(A) = \alpha$ via the isomorphism $h : A \rightleftarrows \alpha$. Let $Y = h[X] \subseteq \alpha$. Since type$(X) =$ type(Y), it is sufficient to show that type$(Y) \leq \alpha$.

Let type$(Y) = \delta$ and via the isomorphism $f : Y \rightleftarrows \delta$. Show by transfinite induction that $f(\gamma) \geq \gamma$ for all $\gamma < \delta$. It then follows that $\gamma \leq f(\gamma) < \alpha$ for all $\gamma < \delta$, and so α is an upper-bound of δ. That is, $\delta \leq \alpha$.

If $Y = \emptyset$, then $\delta = 0 \leq \alpha$. Suppose $Y \neq \emptyset$. Show that $f(\gamma) \geq \gamma$ for all $\gamma < \delta$ by transfinite induction on γ.

$\gamma = 0$. Then $f(0)$ is the least ordinal in Y, so $0 \leq f(0)$.

$\gamma = \eta + 1$. Assume (i.h.) $f(\eta) \geq \eta$. Then $f(\eta + 1)$ is the least element of Y greater than $f(\eta)$, so
$$f(\gamma) = f(\eta + 1) \geq f(\eta) + 1 \geq \eta + 1 = \gamma.$$
γ is a limit. Assume (i.h.) $f(\eta) \geq \eta$ whenever $\eta < \gamma$. Then $f(\gamma)$ is the least ordinal in Y greater than each $f(\eta)$ for $\eta < \gamma$. So,
$$\begin{aligned} f(\gamma) &\geq \sup\{f(\eta) \mid \eta < \gamma\} \\ &\geq \sup\{\eta \mid \eta < \gamma\} \\ &= \gamma. \end{aligned}$$

□

Exercise 6. *Show that $\alpha \cdot \beta =$ type$(\beta \otimes \alpha)$ for all α, β, where $\beta \otimes \alpha$ is the ordered set $\big(\beta \times \alpha, \triangleleft\big)$ and $\triangleleft$ is lexicographic ordering (see Lecture 22, slide 28).*

Proof. The proof is by transfinite induction on β.

$\beta = 0$. Then $0 \times \alpha = 0$, so $\text{type}(0 \otimes \alpha) = 0 = \alpha \cdot \beta$.

$\beta = \gamma + 1$. Suppose (i.h.) $\alpha \cdot \gamma = \text{type}(\gamma \otimes \alpha)$. The ordering $(\gamma+1) \otimes \alpha$ places a copy of α (pairs of the form (γ, δ) where $\delta \in \alpha$) above $\gamma \otimes \alpha$. So,

$$\beta \otimes \alpha = (\gamma + 1) \otimes \alpha = (\gamma \otimes \alpha) \oplus \alpha.$$

Thus,

$$\begin{aligned} \text{type}(\beta \otimes \alpha) &= \text{type}((\gamma \otimes \alpha) \oplus \alpha) \\ &= \text{type}(\gamma \otimes \alpha) \oplus \text{type}(\alpha) \\ &= \alpha \cdot \gamma + \alpha \\ &= \alpha \cdot (\gamma + 1) \\ &= \alpha \cdot \beta. \end{aligned}$$

β is a limit. By the (i.h.), $\alpha \cdot \gamma = \text{type}(\gamma \otimes \alpha)$ whenever $\gamma < \alpha$.

Note that if $\delta < \gamma$ then $\delta \otimes \alpha$ is an initial segment of $\gamma \otimes \alpha$ by the lexicographic ordering. Since β is a limit

$$\beta \otimes \alpha = \bigcup_{\gamma < \beta} \gamma \otimes \alpha.$$

By Lecture 22, slide 16,

$$\begin{aligned} \text{type}(\beta \otimes \alpha) &= \sup\{\text{type}(\gamma \otimes \alpha) \,\big|\, \gamma < \beta\} \\ &= \sup\{\alpha \cdot \gamma \,\big|\, \gamma < \beta\} \\ &= \alpha \cdot \beta. \end{aligned}$$

□

Exercise 7. *Let α be a limit ordinal. Show the following are equivalent (such α are called indecomposable ordinals):*

(a) $\forall \beta < \alpha \, \big(\beta + \alpha = \alpha\big)$.

(b) $\forall \beta, \gamma < \alpha \, \big(\beta + \gamma < \alpha\big)$.

(c) $\exists \delta \, \big(\alpha = \omega^\delta\big)$.

(d) $\forall X \subseteq \alpha \, \big(\mathit{type}(X) = \alpha \;\vee\; \mathit{type}(\alpha - X) = \alpha\big)$.

Hint. (b) $\Rightarrow$ (c): Use the bracketing theorem (Lecture 20, slide 13) with the function $\big(\delta \mapsto \omega^\delta\big)$. For (c) $\Rightarrow$ (d). Prove by transfinite induction on δ. Consider X and X^c where $X^c = \omega^\delta - X$. For the successor *case, where $\omega^\delta = \omega^{\gamma+1}$, break ω^δ into a disjoint sequence of intervals of length ω^γ and consider the types of X and X^c restricted on these intervals. For the limit case, $\omega^\delta = \sup\{\omega^\xi \mid \xi < \delta\}$, so take the disjoint intervals $[\omega^\xi, \omega^{\xi+1})$ and consider the types of X and X^c when restricted to each of these intervals. In each case the intervals are disjoint and unbounded in ω^δ, that is for each interval there is an interval above it. The inductive hypothesis applies to each of these intervals. Use these intervals to show that the type of X or X^c must be ω^δ.*

Proof. (*a*) $\Rightarrow$ (*b*). If $\gamma < \alpha$ then $\beta + \gamma < \beta + \alpha = \alpha$ by the order property of addition, and (a) (the last equality).

(*b*) $\Rightarrow$ (*c*). Consider the function $F_\omega(\delta) = \omega^\delta$. F is normal (Lecture 21, slide 12). By the Bracketing Theorem (Lecture 20, slide 13) there is a unique δ with

$$\omega^\delta \leq \alpha < \omega^{\delta+1} = \omega^\delta + \omega.$$

Suppose $\omega^\delta < \alpha$. Then

$$\omega^\delta \cdot n < \alpha \leq \omega^\delta \cdot (n+1) \qquad \text{for some } n > 1$$

By the division algorithm

$$\alpha = \omega^\delta \cdot n + \gamma \qquad \gamma \leq \omega^\delta.$$

However

$$\omega^\delta \cdot n < \alpha \quad \text{and} \quad \gamma \leq \omega^\delta < \alpha$$

contradicting the assumption that (b).

(*c*) $\Rightarrow$ (*d*). The proof is by transfinite induction on δ. When $\delta = 0$, $\omega^0 = 1$, so for any $X \subseteq 1$, one of $\text{type}(X) = 1$ or $\text{type}(1 - X) = 1$ (the latter only when $X = \emptyset$).

Successor. Suppose that for any $X \subseteq \omega^\delta$, either $\text{type}(X) = \omega^\delta$ or $\text{type}(\omega^\delta - X) = \omega^\delta$. Since $\omega^{\gamma+1} = \omega^\gamma \cdot \omega$, we can break-up $\omega^{\gamma+1}$ into disjoint intervals $W_n = [\omega^n, \omega^{n+1})$, where $\text{type}(W_n) = \omega^\delta$ and

$$\omega^{\delta+1} = \{0\} \cup \bigcup_n W_n.$$

Fix $X \subseteq \omega^{\delta+1}$ and let $Y = \omega^{\delta+1} - X$. For each n, either $\text{type}(W_n - X) = \omega^\delta$ or $\text{type}(W_n - Y) = \omega^\delta$. Suppose there is an unbounded sequence $\{n_i \mid i \in \omega\}$ with $\text{type}(X \cap W_{n_i}) = \omega^\delta$. (If this is not true of X, then it is true of Y by the inductive hypothesis, so we can replace X with Y in the argument.) Let $W = \bigcup_i \big(W_{n_i} \cap X\big) \subset X$, so

$$\omega^{\delta+1} \geq \text{type}(X) \geq \text{type}(W) \geq \omega^\delta \cdot \omega = \omega^{\delta+1}.$$

Thus, $\text{type}(X) = \omega^\delta + 1$.

Limit. Assume δ is a limit ordinal. The argument is very similar as the successor case. Since $\omega^\delta = \sup\{\omega^\xi \mid \xi < \delta\}$, let $W_\xi = [\omega^\xi, \omega^{\xi+1})$. Then, $\text{type}(W_\xi) = \omega^{\xi+1}$ and

$$\omega^\delta = \{0\} \cup \bigcup_{\xi<\delta} W_\xi.$$

Fix $X \subseteq \omega^\delta$ and let $Y = \omega^\delta - X$. For each ξ, either $\text{type}(W_\xi - X) = \omega^{\xi+1}$ or $\text{type}(W_\xi - Y) = \omega^{\xi+1}$. Suppose there is an unbounded sequence in δ, $\{\xi_\eta \mid \eta < \nu\}$, with $\text{type}(X \cap W_{\xi_\eta}) = \omega^{\xi_\eta+1}$. (If this is not true of X, then it is true of Y by the inductive hypothesis and the fact that δ is a limit ordinal, so we can replace X with Y in the argument.) Since $\{\xi_\eta \mid \eta < \nu\}$ is unbounded in δ,

$$\omega^\delta = \sup\{\omega^\xi \mid \xi < \delta\} = \sup\{\omega^{\xi_\eta+1} \mid \eta < \nu\} = \sup\{\text{type}(W_{\xi_\eta} \cap X) \mid \eta < \nu\}$$

Let $W = \bigcup_{\eta<\nu} \big(W_{\xi_\eta} \cap X\big) \subset X$, so

$$\text{type}(W) \geq \sup\{\text{type}(W_{\xi_\eta} \cap X) \mid \eta < \nu\} = \omega^\delta.$$

Putting this together,

$$\omega^\delta \geq \text{type}(X) \geq \text{type}(W) \geq \omega^\delta$$

Thus, $\text{type}(X) = \omega^\delta$.

$(d) \Rightarrow (a)$. Fix $\beta < \alpha$. Let $X = \beta < \alpha$. Then, $\text{type}(X) = \beta < \alpha$, so $\text{type}(\alpha - X) = \alpha$. But

$$\alpha = \text{type}(X) + \text{type}(X - \alpha) = \beta + \alpha.$$

□

Exercise 8. *Prove that the following combinatorial definition of ordinal exponentiation is equivalent to the definition by transfinite recursion. Let*

$$A(\alpha, \beta) = \big\{f \in {}^\beta\alpha \mid \{\xi \mid f(\xi) \neq 0\} \text{ is finite } \big\}.$$

For $f, g \in A(\alpha, \beta)$ and $f \neq g$, say $f \lhd g$ iff $f(\xi) < g(\xi)$, where ξ is the largest ordinal such that $f(\xi) \neq g(\xi)$. Show that $\alpha^\beta = \text{type}\big(A(\alpha, \beta), \lhd\big)$.

Proof. For this exercise only, I will write $\text{dom}(f)$ to mean the finite set $\{\xi \in \beta \mid f(\xi) > 0\}$.

The proof that $\big(A(\alpha, \beta), \lhd\big)$ is a well-ordered set and that $\alpha^\beta = \text{type}\big(A(\alpha, \beta), \lhd\big)$ is proven simultaneously is by transfinite recursion on β. When $\beta = 0$, $A(\alpha, \beta) = \{\emptyset\}$ and $\alpha^\beta = 1$, so $\alpha^\beta = \text{type}\big(A(\alpha, \beta), \lhd\big)$.

If $\beta = \gamma + 1$, then $\alpha^\gamma = \text{type}\big(A(\alpha, \gamma), \lhd\big)$; however, $A(\alpha, \beta) \cong \alpha \times A(\alpha, \gamma)$ by the map $f \mapsto \big(f(\gamma), f \restriction \gamma\big)$ and where we take $\lhd$ as the order on the first set, and lexicographic order on the second. By the inductive hypothesis and Exercise 6

$$\alpha^\beta = \alpha^\gamma \cdot \alpha = \text{type}\big(\alpha \times A(\alpha,\gamma), \lhd\big).$$

If β is a limit, then by the inductive hypothesis for any $\gamma < \beta$,

$$\alpha^\gamma = \text{type}\big(A(\alpha,\gamma), \lhd\big).$$

Note that when $\eta < \gamma$, $A(\alpha,\eta)$ is an initial segment of $A(\alpha,\gamma)$, and that $A(\alpha,\beta) = \bigcup_{\gamma<\beta} A(\alpha,\gamma)$, since $\mathrm{dom}(f)$ is finite for $f \in A(\alpha,\beta)$ and so $f \in A(\alpha,\gamma)$ for any $\gamma > \max \mathrm{dom}(f)$. By Lecture 22, slide 16 and the i.h.

$$\begin{aligned} \text{type}\big(A(\alpha,\beta), \lhd\big) &= \sup_{\gamma<\beta} \text{type}\big(A(\alpha,\gamma), \lhd\big) \\ &= \sup_{\gamma<\beta} \alpha^\gamma \\ &= \alpha^\beta. \end{aligned}$$

$\square$