MATH 582 HOMEWORK 4
WEEK 8

Winter, 2009
Due March 13

Exercise 1. *A set z is a transitive set if and only if $\bigcup z \subseteq z$.*

Proof. ($\Rightarrow$). Suppose z is a transitive set. If $x \in \bigcup z$, then $x \in y$ for some $y \in z$. But $y \subseteq z$ (transitive set), so $x \in z$. Since x was arbitrary, $\bigcup z \subseteq z$.

($\Leftarrow$). Suppose $\bigcup z \subseteq z$. Let $y \in z$, and fix $x \in y$. Then $x \in \bigcup z \subseteq z$, so $x \in z$. Since x was arbitrary, $y \subseteq z$. Thus, z is a transitive set. □

Exercise 2. *Give an example of sets x and y satisfying the following: y is a transitive set and $x \in y$ but x is not a transitive set.*

Proof. Let $y = \{0, 1, \{1\}\}$ and $x = \{1\}$. y is transitive since $0, 1, \{1\} \subseteq y$, but x is not transitive, since $1 \in x$ but $1 = \{0\} \nsubseteq \{1\} = x$. □

Exercise 3. *If X is a nonempty set of ordinals, then $\bigcap X$ is an ordinal, and the least element of X.*

Proof. Let X be a nonempty set of ordinals. We showed that the intersection of two ordinals is an ordinal (Lecture 17, Little Lemma 2 on slide 23). Of course, we require X to be nonempty, so that $\bigcap X$ is a set.

Since X is a nonempty set of ordinals, there is a least member $\beta \in X$. I'll show $\beta = \bigcap X$. First, $\bigcap X \subseteq \beta$ by the definition of intersection. Next, let $\xi \in \beta$, then $\xi \in \alpha$ for every $\alpha \in X$ since $\beta \leq \alpha$ for every $\alpha \in X$. So, $\xi \in \bigcap X$ and thus, $\beta \subseteq \bigcap X$. Therefore, $\beta = \bigcap X$.

□

Exercise 4. *Let $X \subset$ ON.*

(a) $\bigcup X$ is an ordinal.
(b) $\bigcup X$ is an upper bound for X: for all $\alpha \in X$, $\alpha \leq \bigcup X$.
(c) $\bigcup X$ is the least upper bound for X: for all ordinal γ, if γ is an upper bound for X, then $\bigcup X \leq \gamma$.
(d) Suppose X is nonempty and has no greatest ordinal. Show $\bigcup X$ is a limit ordinal.

(e) *Show: There is an ordinal α such that $\alpha > \beta$ for all $\beta \in X$. (So, $\alpha \notin X$, and this provides another proof that there is no set X containing all ordinals.)*

Proof. (a). It must be shown that $\in$ well-order $\bigcup X$ and $\bigcup X$ is a transitive set. If $X = \emptyset$, then $\bigcup X = \emptyset$ which is an ordinal. So, suppose $X \neq \emptyset$.

First, since $X \subseteq \mathbf{ON}$ and $\mathbf{ON}$ is well-ordered by membership, this is true for X as well. (See Lecture 12, slide 20 – the fact that $\mathbf{ON}$ is a proper class is irrelevant in the proof.)

Next, I will prove $\bigcup X$ is a transitive set. Note that if X is any family of sets (not just ordinals) and $y \in X$ then $y \subseteq \bigcup X$ (this follows directly from the definition of union). Now, since X is a family of ordinals, for any $\beta \in \bigcup X$, there is some $\alpha \in X$ with $\beta \in \alpha$, so that $\beta \subseteq \alpha \subseteq \bigcup X$. Thus, $\beta \subseteq \bigcup X$.

(b). From (a), it was shown that for $\alpha \in X$, $\alpha \subseteq \bigcup X$. But, $\bigcup X$ is an ordinal, so $\alpha \in \bigcup X$ or $\alpha = \bigcup X$. That is, $\alpha \leq X$.

(c). Let γ be an upper bound for X, so that $\alpha \leq \gamma$ for each $\alpha \in X$. Let $\beta \in \bigcup X$ and $\alpha \in X$ with $\beta \in \alpha$. But, $\alpha \in \gamma$ implies $\alpha \subseteq \gamma$, so that $\beta \in \gamma$. Thus, $\bigcup X \subseteq \gamma$; that is, $\bigcup X \leq \gamma$.

(d). Suppose $X \subseteq ON$, but has no greatest ordinal. I'll show that $\bigcup X$ also has no greatest ordinal, so must be a limit ordinal. Let $\gamma < \bigcup X$, that is, $\gamma \in \bigcup X$. So, for some $\alpha \in X$, $\gamma \in \alpha$. But α is not greatest in X, so let $\alpha < \beta \in X$. Thus, $\gamma < \alpha \in \bigcup X$.

(e). Let X be a subset of ordinals. $S(\bigcup X)$ is bigger than any ordinal in X since $\bigcup X < S(\bigcup X)$. (See Lecture 17, slide 26.)

Note. This is not part of the proof. When X has a largest element α, then $\bigcup X = \alpha$: if $\gamma \in \bigcup X$ then $\gamma \in \beta \in X$ for some β; but since α is greatest in X, $\gamma \in \alpha$, so $\bigcup X \leq \alpha$. On the other hand, $\alpha \leq \bigcup X$ by part b, so $\alpha = \bigcup X$.

If X has no greatest element, then $\bigcup X$ is greater than any ordinal in X: If $\alpha \in X$, there is some $\beta \in X$ with $\alpha < \beta \leq \bigcup X$ (the last by part b).

□

Exercise 5. *If x is a finite ordinal then so is every $y \in x$, and $S(x)$ is also a finite ordinal. Furthermore, every natural number is a finite ordinal.*

Proof. □