MATH 582 HOMEWORK 4
WEEK 7

Winter, 2009

Due March 13

Exercise 1. *Recall the definition of multiplication from HW3 on Week 6 (Exercise 2). Prove that multiplication respects order: for every* m, n, k

$$k > 0 \rightarrow \Big(m < n \leftrightarrow m \cdot k < n \cdot k\Big).$$

Hint. I have not proven all the intermediate lemmas you will need.

Exercise 2. *Let* $(W, \prec)$ *be a nonempty, totally ordered set. For* $p, q \in W$ *we say that* q *is a* successor *of* p *if* $p \prec q$ *and there is no* $r \in W$ *with* $p \prec r \prec q$. *Assume that* $(W, \prec)$ *has the following additional properties*

(a) Every $p \in W$ *has a successor.*

(b) Every nonempty subset of W *has a* $\prec$*-least element.*

(c) If $p \in W$ *is not the* $\prec$*-least element of* W*, then* p *is a successor of some* $q \in W$*.*

Prove that $(W, \prec)$ *is isomorphic to* $(\mathbb{N}, <)$*. Show that the conclusion does not hold if one of the conditions (a)-(c) is omitted.*

Exercise 3. *Prove: If* $X \subseteq \mathbb{N}$ *has no upper bound in* $\mathbb{N}$*, then there is a bijection* $f : X \rightleftarrows \mathbb{N}$.

Exercise 4. *Let* $A = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots\big\}$ *and the function* $T : A \to A$ *be defined by* $S^* = \big\{(n, \{n\}) \,\big|\, n \in A\big\}$*. Show that* $(A, \emptyset, T)$ *is a system of natural numbers (that is, satifies the Dedekind-Peano axioms.)*

Hint. *You will need to prove* A *exists, and this requires some thought about what "…" means. You will find the Complete Recursion Theorem in Lecture 16, slide 16, useful.*

Exercise 5. *Think about, but do not hand in this exercise. Modify our proof of the Primitive Recursion Theorem (Lecture 14, slide 10) to produce a proof of the Complete Recursion Theorem (Lecture 16, slide 16).*