MATH 582 HOMEWORK 4
WEEK 7

Winter, 2009
Due March 13

Exercise 1. *Recall the definition of multiplication from HW3 on Week 6 (Exercise 2). Prove that multiplication respects order: for every* m, n, k

$$k > 0 \rightarrow \Big(m < n \leftrightarrow m \cdot k < n \cdot k\Big).$$

Hint. I have not proven all the intermediate lemmas you will need.

Proof. We will need to prove that the successor operator and addition preserve order.

(i). Successor preserves order: for all m and n

$$m < n \leftrightarrow S(m) < S(n).$$

Suppose $m < n$. Then there exists a $k > 0$ such that $n = m + k$. Since $k > 0$, there is an j with $k = S(j)$, so $n = m + S(j)$. Thus, $S(n) = S(m + S(j)) = S(m) + S(j)$, and so $S(m) < S(n)$.

Suppose $S(m) < S(n)$. Then, $m < S(m) \leq n$, so $m < n$.

(ii). Addition preserves order: for all m, n, k

$$m < n \leftrightarrow m + k < n + k.$$

The proof is by induction on k. For the basis case, $k = 0$, so for any m and n

$$m < n \leftrightarrow m + 0 < n + 0$$

since $m = 0 = m$ and $n + 0 = n$.

Inductive case. Suppose the property holds for k: for any m and n

$$m < n \leftrightarrow m + k < n + k.$$

Suppose $m < n$. Then $m+k < n+k$ by the i.h., so $S(m+k) < S(n+k)$ by (i), and thus $m+S(k) < n+S(k)$ by the definition of addition. Suppose $m+S(k) < n+S(k)$. Then $S(m+k) < S(n+k)$, so $m+k < n+k$ by (i), and $m < n$ by the i.h.

Main proof. The proof is by induction on k. The basis case is trivial, since when $k = 0$ the antecedent is false. So, assume for the inductive hypothesis that for each m and n

$$k > 0 \rightarrow \Big(m < n \leftrightarrow m \cdot k < n \cdot k\Big).$$

Note that $S(k) > 0$, so the antecedent holds for $S(k)$, but it might not hold for k. If $k = 0$, then $S(k) = S(0) = 1$. Since $p \cdot 1 = p$ for any p, it follows for any m and n

$$m < n \;\leftrightarrow\; m \cdot 1 < n \cdot 1.$$

Suppose $k > 0$. If $m < n$, then, using the i.h. and part (a),

$$\begin{array}{rcll} m \cdot S(k) & = & m \cdot k + m & \\ & < & m \cdot k + n & \text{(ii)} \\ & < & n \cdot k + n & \text{i.h. and (ii)} \\ & = & n \cdot S(k). & \end{array}$$

If $m \not< n$, then either $m = n$, so that $m \cdot S(k) = n \cdot S(k)$, or $n < m$, so that $n \cdot S(k) < m \cdot S(k)$. In either case, $m \cdot S(k) \not< n \cdot S(k)$.

□

Exercise 2. *Let $(W, \prec)$ be a nonempty, totally ordered set. For $p, q \in W$ we say that q is a* successor *of p if $p \prec q$ and there is no $r \in W$ with $p \prec r \prec q$. Assume that $(W, \prec)$ has the following additional properties*

(a) Every $p \in W$ has a successor.

(b) Every nonempty subset of W has a $\prec$-least element.

(c) If $p \in W$ is not the $\prec$-least element of W, then p is a successor of some $q \in W$.

Prove that $(W, \prec)$ is isomorphic to $(\mathbb{N}, <)$. Show that the conclusion does not hold if one of the conditions (a)-(c) is omitted.

Proof. Let $(W, \prec)$ be a totally ordered set satisfying (a)-(c). Note that every element $p \in W$ has a *unique* successor. By (a) every element has a successor; by the total order, if $p \prec q$ and $p \prec r$ then $q \prec r$ or $r \prec q$ or $q = r$, and so q and r cannot be distinct successors of p.

You could define a function $\phi : \mathbb{N} \to W$ by recursion. However, in Lecture 16, slide 6, I proved that if $(W, \prec)$ satisfies the three conditions

(i) There is no largest element:
for every $w \in W$, there is a $z \in W$ with $w \prec z$.

(ii) Every nonempty subset of W has a $\prec$-least element.

(iii) Every nonempty subset of W that has an upper bound has a $\prec$-greatest element.

is isomorphic to $(\mathbb{N}, \prec)$. (See also H+J, Theorem 3.3.4, page 49, for the same result.) The only condition that remains to be shown is (iii).

Let $X \subseteq W$ be a nonempty set, and suppose $p \in W$ is an upper bound of X. Let

$$U = \{q \in W \mid q \text{ an upper bound of } X\},$$

so $p \in U$, and by condition (b), U has a $\prec$-least element, a. I will show that $a \in X$ – if this is true, then a is the $\prec$-greatest element since $x \prec a$ for all $x \in X - \{a\}$. Suppose $a \notin X$. Let $b \in W$ be such that a is the successor of b – such a b exists by condition (c), since a cannot be $\prec$-least, as $X \neq \emptyset$. I will show that b is also an upper bound on X. Suppose $x \in X$, so $x \prec a$. Either $x = b$ or $x \prec b$, since $b \prec x$ implies $b \prec x \prec a$, contradicting condition (c). Thus, b is an upper bound of X and $b \prec a$, which contradicts the choice of a. Therefore, $a \in X$.

Here are the counterexamples.
(a). Let $W = \{0\}$ with the empty relation $\prec = \emptyset$. Then (b) and (c) are satisfied since 0 is $\prec$-least and $\prec$-greatest in W. And (a) fails, since 0 has no successor.

(b). Consider the integers $(\mathbb{Z}, <)$. (a) holds since $x+1$ is the successor of x for every integer x. (c) holds: every nonempty subset $X \subset \mathbb{Z}$ which is bounded below, has a least element, so the same argument that (c) holds for $\mathbb{N}$ works for $\mathbb{Z}$ as well. This was proven in Lecture 16, Slide 3. (b) fails: $\mathbb{Z}$ has no smallest element.

(c). Let $W = \omega + \omega$ with the usual ordering $< = \in$. (a) holds since the successor of $\alpha \in W$ is $\alpha + 1 \in W$. (b) holds since every ordinal is well-ordered by $<$. (c) fails since the set $\omega = \{\alpha \mid \alpha < \omega\}$ is bounded by ω, but has no $<$-greatest element.

□

Exercise 3. *Prove: If $X \subseteq \mathbb{N}$ has no upper bound in $\mathbb{N}$, then there is a bijection $f : X \rightleftarrows \mathbb{N}$.*

Proof. You could define the bijection by recursion, but I will use the previous exercise instead. Let $(X, \prec)$ be the ordered set where $\prec = < \restriction \mathbb{N} \times \mathbb{N}$. Then $(X, \prec)$ is well-ordered (see Lecture 12, slide 20).

It remains to show conditions (a) and (c) hold. For (c), if $Y \subseteq X$ has a $\prec$-upper bound, then it has a $<$-upper bound in $\mathbb{N}$ (these need not be the same). Since (c) holds for $\mathbb{N}$, Y has a $<$-greatest element, which must also be $\prec$-greatest. For (a), fix any $n \in X$, so the set $\{n\}$ has a $\prec$-upper bound, by our assumption that X is unbounded in $\mathbb{N}$. Let $m \in X$ be the $\prec$-least upper bound. Then, $n \prec m$ and there can be no $p \in X$ with $n \prec p \prec m$, since m is $\prec$-least above n. Thus, m is a successor of n. □

Exercise 4. *Let* $A = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \dots\big\}$ *and the function* $T : A \to A$ *be defined by* $T = \big\{(n, \{n\}) \,\big|\, n \in A\big\}$*. Show that* $(A, \emptyset, T)$ *is a system of natural numbers (that is, satifies the Dedekind-Peano axioms.)*

Hint. *You will need to prove A exists, and this requires some thought about what "..." means. You will find the Complete Recursion Theorem in Lecture 16, slide 16, useful.*

Proof. Let $\mathbf{G} : \mathbf{V} \to \mathbf{V}$ be the class function defined by $\mathbf{G}(x) = \{x\}$. Note that $\mathbf{G}(x) = \mathbf{G}(y)$ iff $x = y$, by Extensionality.

By Complete Recursion (Lecture 16) there is a class function $\mathbf{F} : \mathbb{N} \to \mathbf{V}$ satisfying

$$\begin{aligned} \mathbf{F}(0) &= \emptyset \\ \mathbf{F}(S(n)) &= \{\mathbf{F}(n)\}. \end{aligned}$$

By Replacement and Comprehension, there is a set function $f : \mathbb{N} \to A$, where $A = \mathrm{ran}(\mathbf{F})$.

If we define $T : A \to A$ by $T = \big\{(n, \{n\}) \,\big|\, n \in A\big\}$, then

$$f(S(n)) = T(f(n)).$$

So, $f : \mathbb{N} \to A$ is a bijection which preserves successors, so $(A, \emptyset, T)$ is a natural number system (that is, satisfies the five Dedekind-Peano Axioms (Lecture 13, slide 3).

□

Exercise 5. *Think about, but do not hand in this exercise. Modify our proof of the Primitive Recursion Theorem (Lecture 14, slide 10) to produce a proof of the Complete Recursion Theorem (Lecture 16, slide 16).*