MATH 582 HOMEWORK 3
WEEK 6

Winter, 2009
Due February 20

Exercise 1. *Prove the following:*

Suppose $(\mathbb{N}_1, 0_1, S_1)$ *and* $(\mathbb{N}_2, 0_2, S_2)$ *are two systems of natural numbers, where* $+_1, +_2$ *are their respective canonical operations of addition. Then the canonical isomorphism* $\pi : \mathbb{N}_1 \rightleftarrows \mathbb{N}_2$ *respects this operation: for all* $n, m \in \mathbb{N}_1$,

$$\pi(n +_1 m) \quad = \quad \pi(n) +_2 \pi(m)$$

Exercise 2. *Define multiplication recursively for all* $m, n \in \mathbb{N}$ *by*

$$\begin{aligned} m \cdot 0 &= 0 \\ m \cdot (n+1) &= m \cdot n + m \end{aligned}$$

Prove $\cdot$ *satisfies the following properties: for all* $m, n, k \in \mathbb{N}$,
(a) (Associativity) $(m \cdot n) \cdot k = m \cdot (n \cdot k)$,
(b) (Commutativity) $m \cdot n = n \cdot m$.
(c) (Distributivity) $m \cdot (n + k) = m \cdot n + m \cdot k$.

Exercise 3. *The version of the Recursion Theorem we have proven in class (including the parameterized version) does not directly allow definitions such as the factorial function:*

$$\begin{aligned} 0! &= 1 \\ S(n)! &= n! \cdot S(n) \end{aligned}$$

since, the function defined by recursion does not have access to the variable of recursion n.

Prove the following version of the Recursion Theorem which remedies this deficiency:

For every

$$a \in E \qquad h : E \times \mathbb{N} \to E$$

there exists a unique $f : \mathbb{N} \to E$ *satisfying*

$$\begin{aligned} f(0) &= a \\ f(S(n)) &= h\big(f(n), n\big) \qquad n \in \mathbb{N} \end{aligned}$$

Hint. *First, define a function* $F : \mathbb{N} \to E \times \mathbb{N}$ *by the Recursion Theorem which "carries along" its argument: let* $a' = (a, 0)$ *and define* $h' : E \times \mathbb{N} \to E \times \mathbb{N}$ *by* $h'\big(e, n\big) = (h(e, n), S(n))$. *Now,* f *simply peels-off the first coordinate of* F.