MATH 582 HOMEWORK 3
WEEK 6

Winter, 2009

Due February 20

Exercise 1. *Prove the following:*

Suppose $(\mathbb{N}_1, 0_1, S_1)$ *and* $(\mathbb{N}_2, 0_2, S_2)$ *are two systems of natural numbers, where* $+_1, +_2$ *are their respective canonical operations of addition. Then the canonical isomorphism* $\pi : \mathbb{N}_1 \rightleftarrows \mathbb{N}_2$ *respects this operation: for all* $n, m \in \mathbb{N}_1$,

$$\pi(n +_1 m) \quad = \quad \pi(n) +_2 \pi(m)$$

Exercise 2. *Define multiplication recursively for all* $m, n \in \mathbb{N}$ *by*

$$\begin{aligned} m \cdot 0 &= 0 \\ m \cdot (n+1) &= m \cdot n + m \end{aligned}$$

Prove $\cdot$ *satisfies the following properties: for all* $m, n, k \in \mathbb{N}$,

(a) (Associativity) $(m \cdot n) \cdot k = m \cdot (n \cdot k)$,

(b) (Commutativity) $m \cdot n = n \cdot m$.

(c) (Distributivity) $m \cdot (n + k) = m \cdot n + m \cdot k$.

Proof. The proof of Associativity and Commutativity are very similar to those same properties for addition (from Lecture 15). I will do Distributivity here. I will prove the following by induction on m, for every n and k

$$(n + k) \cdot m = n \cdot m + k \cdot m.$$

The version in the Exercise follows by Commutativity of multiplication.

Basis case. When $m = 0$, for every n and k

$$(n + k) \cdot = 0 = 0 + 0 = n \cdot 0 + k \cdot 0.$$

Induction case. Suppose Distributivity holds for m:

$$(n + k) \cdot m = n \cdot m + k \cdot m \qquad \text{for all } n \text{ and } k.$$

Then, for any n and k

$$\begin{aligned} (n + k) \cdot S(m) &= (n + k) \cdot m + (n + k) \\ &= n \cdot m + k \cdot m + n + k \\ &= \big(n \cdot m + n\big) + \big(k \cdot m + k\big) \\ &= n \cdot S(m) + m \cdot S(m). \end{aligned}$$

$\square$

Exercise 3. *The version of the Recursion Theorem we have proven in class (including the parameterized version) does not directly allow definitions such as the factorial function:*

$$\begin{aligned} 0! &= 1 \\ S(n)! &= n! \cdot S(n) \end{aligned}$$

since, the function defined by recursion does not have access to the variable of recursion n.

Prove the following version of the Recursion Theorem which remedies this deficiency:

> *For every*
> $$a \in E \qquad h : E \times \mathbb{N} \to E$$
> *there exists a unique $f : \mathbb{N} \to E$ satisfying*
> $$\begin{aligned} f(0) &= a \\ f(S(n)) &= h\big(f(n), n\big) \qquad n \in \mathbb{N} \end{aligned}$$

***Hint**. First, define a function $F : \mathbb{N} \to E \times \mathbb{N}$ by the Recursion Theorem which "carries along" its argument: let $a' = (a, 0)$ and define $h' : E \times \mathbb{N} \to E \times \mathbb{N}$ by $h'\big(e, n\big) = (h(e, n), S(n))$. Now, f simply peels-off the first coordinate of F.*

Proof. Let $a \in E$ and $h : E \times \mathbb{N} \to E$. Define $H : E \times \mathbb{N} \to E \times \mathbb{N}$ by

$$H(e, n) = \big(h(e, n), S(n)\big)$$

Define $F : \mathbb{N} \to E \times \mathbb{N}$ by primitive recursion

$$\begin{aligned} F(0) &= (a, 0) \\ F(S(n)) &= H(F(n)) = \big(h(e, n), S(n)\big) \end{aligned}$$

Let $\pi_1 : E \times \mathbb{N} \to E$ be $\pi_1(e, n) = e$. Define $f : \mathbb{N} \to E$ by

$$f(n) = \pi_1\big(F(n)\big).$$

So,

$$\begin{aligned} f(0) &= \pi_1\big(F(0)\big) = \pi_1\big((a, 0)\big) = a \\ f(S(n)) &= \pi_1\big(F(S(n))\big) = \pi_1\big(h(f(n), n), S(n)\big) = h\big(f(n), n\big). \end{aligned}$$

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