MATH 582 HOMEWORK 3
WEEK 5

Winter, 2009
Due February 20

Exercise 1. *Let* $(A, <)$ *and* $(B, \prec)$ *be ordered sets with* $A \cap B = \emptyset$. *Define* $\lhd$ *on* $A \cup B$ *as follows: for any* $x, y \in A \cup B$ *let* $x \lhd y$ *iff either*

(i) $x, y \in A$ *and* $x < y$, *or*
(ii) $x, y \in B$ *and* $x < y$, *or*
(iii) $x \in A$ *and* $y \in B$

(The relation $\lhd$ *puts everything in* A *before* B, *and otherwise respects the ordering* $<$ *on* A *and* $\prec$ *on* B.*)*

Prove.

(a) $(A \cup B, \lhd)$ *is an ordered set.*
(b) $(A \cup B, \lhd)$ *is a totally ordered set, when* $(A, <)$ *and* $(B, \prec)$ *are totally ordered.*
(c) $(A \cup B, \lhd)$ *is a well-ordered set, when* $(A, <)$ *and* $(B, \prec)$ *are well-ordered.*

Exercise 2. *Let* $(A, <)$ *and* $(B, \prec)$ *be ordered sets. The lexicographic product on* $A \times B$ *is the relation* $\lhd$ *on* $A \times B$ *defined by*

$$(a, b) \lhd (a', b') \quad \leftrightarrow \quad a < a' \vee (a = a' \wedge b \prec b')$$

Prove.

(a) $(A \times B, \lhd)$ *is an ordered set.*
(b) $(A \times B, \lhd)$ *is a totally ordered set, when* $(A, <)$ *and* $(B, \prec)$ *are totally ordered.*
(c) $(A \times B, \lhd)$ *is a well-ordered set, when* $(A, <)$ *and* $(B, \prec)$ *are well-ordered.*

Note. We can view $A \times B$ *as two-letter words whose first letter comes from* A *and whose second letter comes from* B. *Then* $\lhd$ *is the dictionary order of these two-letter words: use the first letter to order elements, and the second letter to break ties.*

Exercise 3. *In this exercise you will show that the domain and range of a relation exist, which is independent of the specific definition of "ordered pair". Suppose we have defined "ordered pair" in some way* $[(x, y)]$ *(as a set), and assume that we can prove for all* x, x', y, y'

$$[(x, y)] = [(x', y')] \quad \rightarrow \quad x = x' \wedge y = y'.$$

Prove that the following sets exist for all sets R:

$$\{x \mid \exists y\big([(x,y)] \in R\big)\}$$
$$\{y \mid \exists x\big([(x,y)] \in R\big)\}$$

(Hint: use Replacement and Comprehension.)

Exercise 4. *Functions f and g are compatible if $f(x) = g(x)$ for all $x \in dom(f) \cap dom(g)$. A family of functions $\mathcal{F}$ is a compatible family of functions if any two functions $f, g \in \mathcal{F}$ are compatible.*

Prove.

(a) Let f, g be functions. Then f and g are compatible iff $f \cup g$ is a function.

(b) Let $\mathcal{F}$ be a family of functions. Then $\mathcal{F}$ is a compatible family of functions iff $\bigcup \mathcal{F}$ is a function with $dom(\bigcup \mathcal{F}) = \bigcup_{f \in \mathcal{F}} dom(f)$.