MATH 582 HOMEWORK 3
WEEK 5

Winter, 2009

Due February 20

Exercise 1. *Let $(A,<)$ and $(B,\prec)$ be ordered sets with $A \cap B = \emptyset$. Define $\lhd$ on $A \cup B$ as follows: for any $x, y \in A \cup B$ let $x \lhd y$ iff either*

(i) $x, y \in A$ and $x < y$, or

(ii) $x, y \in B$ and $x < y$, or

(iii) $x \in A$ and $y \in B$

(The relation $\lhd$ puts everything in A before B, and otherwise respects the ordering $<$ on A and $\prec$ on B.)

***Prove**.*

(a) $(A \cup B, \lhd)$ is an ordered set.

(b) $(A \cup B, \lhd)$ is a totally ordered set, when $(A,<)$ and $(B,\prec)$ are totally ordered.

(c) $(A \cup B, \lhd)$ is a well-ordered set, when $(A,<)$ and $(B,\prec)$ are well-ordered.

Proof. Note the following

- $\lhd \restriction A \times A = <$ and $\lhd \restriction B \times B = \prec$
- if $x \in B$ and $x \lhd y$, then $y \in B$, and
- if $y \in A$ and $x \lhd y$, then $x \in A$.

(a). $\lhd$ is clearly irreflexive, since each of $<$ and $\prec$ are. For transitivity, fix any x, y, z with $x \lhd y$ and $y \lhd z$. If $x \in B$ then each of $x, y, z \in B$, so $x \prec z$ and thus $x \lhd z$. If $z \in A$, then each of $x, y, z \in A$, so $x < z$ and thus $x \lhd z$. Finally, if $x \in A$ and $z \in B$, then $x \lhd z$ by definition.

(b). For trichotomy: fix $x \neq y$. If $x, y \in A$ (or $x, y \in B$) then $x \lhd y$ or $y \lhd x$ by trichotomy for $<$ (or for $\prec$). If $x \in A$ and $y \in B$, then $x \lhd y$ be definition.

(c). For well-ordering: let $X \subseteq A \cup B$ be nonempty. If $X \cap A \neq \emptyset$, then $X \cap A$ has a $<$-least element, which must also be $\lhd$-least in X since every element in $X \cap A$ is $\lhd$-below each element in $X \cap B$. If $X \cap A = \emptyset$, then $X \subseteq B$, so has a $\prec$-least element, which also be $\lhd$-least in X. $\square$

Exercise 2. *Let $(A,<)$ and $(B,\prec)$ be ordered sets. The lexicographic product on $A \times B$ is the relation $\lhd$ on $A \times B$ defined by*

$$(a, b) \lhd (a', b') \quad \leftrightarrow \quad a < a' \vee (a = a' \wedge b \prec b')$$

Prove*.*

(a) $(A \times B, \lhd)$ *is an ordered set.*
(b) $(A \times B, \lhd)$ *is a totally ordered set, when* $(A, <)$ *and* $(B, \prec)$ *are totally ordered.*
(c) $(A \times B, \lhd)$ *is a well-ordered set, when* $(A, <)$ *and* $(B, \prec)$ *are well-ordered.*

Note. We can view $A \times B$ *as two-letter words whose first letter comes from* A *and whose second letter comes from* B*. Then* $\lhd$ *is the dictionary order of these two-letter words: use the first letter to order elements, and the second letter to break ties.*

Proof. (a). $\lhd$ is clearly irreflexive, since each of $<$ and $\prec$ are. For transitivity, fix any elements of $A \times B$ with $(a, b) \lhd (c, d)$ and $(c, d) \lhd (e, f)$. There are several cases.

- If $a = c = e$, then $b \prec d \prec f$ so $b \prec f$, and thus $(a, b) \lhd (e, f)$.
- If $a < c < e$, then $(a, b) \lhd (e, f)$.
- If $a = c < e$ or $a < c = e$, then $a < e$, so $(a, b) \lhd (e, f)$.

(b). For trichotomy, fix $(a, b) \neq (c, d)$. If $a < c$ (or $c < a$) then $(a, b) \lhd (c, d)$ (or $(c, d) \lhd (a, b)$). If $a = c$, then either $b < d$ or $d < b$. In the first case, $(a, b) \lhd (c, d)$, and in the second case $(c, d) \lhd (a, b)$.

(c). For well-ordering, let $X \subseteq A \times B$ be a nonempty set. Let

$$X_1 = \{a \in A \,\big|\, \exists y \in B \, (a, y) \in X\}$$

So, $X_1 \subseteq A$ is nonempty and has a $<$-least element, a.

Define

$$X_2 = \{b \in B \,\big|\, (a, b) \in X\}.$$

So, $X_2 \subseteq B$ is nonempty and has a $\prec$-least element b. Then (a, b) is $\lhd$-least in X: if $(x, y) \in X$, then $a \leq x$ by the choice of a, and if $a = x$, then $b \leq y$ by the choice of b.

□

Exercise 3. *In this exercise you will show that the domain and range of a relation exist, which is independent of the specific definition of "ordered pair". Suppose we have defined "ordered pair" in some way* $[(x, y)]$ *(as a set), and assume that we can prove for all* x, x', y, y'

$$[(x, y)] = [(x', y')] \quad \to \quad x = x' \wedge y = y'.$$

Prove that the following sets exist for all sets R*:*

$$\{x \,\big|\, \exists y \big([(x, y)] \in R\big)\}$$
$$\{y \,\big|\, \exists x \big([(x, y)] \in R\big)\}$$

(Hint: use Replacement and Comprehension.)

Proof. Define $\varphi(u,x)$ be defined by

$$\exists y\, u = [(x,y)].$$

Suppose $u \in R$ and x, x' are such that $\varphi(u,x)$ and $\varphi(u,x')$. Then, $u = [(x,y)]$ and $u = [(x',y')]$ for some y, y'. So, $x = x'$, and thus

$$\forall u \in R \exists! x \varphi(u,x).$$

By Replacement, there is a set B which collects all x with $[(x,y)] \in R$, for some y. Now use Comprehension with B to get

$$\{x \mid \exists y\big([(x,y)] \in R\big)\}.$$

Define $\psi(u,y)$ be defined by

$$\exists x\, u = [(x,y)].$$

Suppose $u \in R$ and y, y' are such that $\psi(u,y)$ and $\psi(u,y')$. Then, $u = [(x,y)]$ and $u = [(x',y')]$ for some x, x'. So, $y = y'$, and thus

$$\forall u \in R \exists! y \psi(u,y).$$

By Replacement, there is a set C which collects all y with $[(x,y)] \in R$, for some x. Now use Comprehension with C to get

$$\{y \mid \exists x\big([(x,y)] \in R\big)\}.$$

□

Exercise 4. *Functions f and g are compatible if $f(x) = g(x)$ for all $x \in dom(f) \cap dom(g)$. A family of functions $\mathcal{F}$ is a compatible family of functions if any two functions $f, g \in \mathcal{F}$ are compatible.*

Prove*.*

(a) Let f, g be functions. Then f and g are compatible iff $f \cup g$ is a function.

(b) Let $\mathcal{F}$ be a family of functions. Then $\mathcal{F}$ is a compatible family of functions iff $\bigcup \mathcal{F}$ is a function with $dom(\bigcup \mathcal{F}) = \bigcup_{f \in \mathcal{F}} dom(f)$.

Proof. (a). Let f, g be functions.

Suppose f and g are compatible. Let $(x,y) \in f$ and $(x,z) \in g$. By compatibility, $y = z$. Since f and g are themselves functions, it follows that $f \cup g$ is a function.

Suppose $f \cup g$ is a function. Let $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$. If $f(x) \neq g(x)$, then $f \cup g$ would not be a function as $(x, f(x)), (x, g(x)) \in f \cup g$.

(b). Let $\mathcal{F}$ be a family of functions.

Suppose $\mathcal{F}$ is a compatible family. Let $(x, y), (x, z) \in \bigcup \mathcal{F}$. So, there are f, g with $(x, y) \in f$ and $(x, z) \in g$. But f, g are compatible, so $y = z$.

Suppose $\bigcup \mathcal{F}$ is a function. Let $f, g \in \mathcal{F}$ and $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$; but, $f(x) = \bigcup \mathcal{F}(x) = g(x)$. So, f, g are compatible. $\square$