MATH 582 HOMEWORK 2
WEEK 3

Winter, 2009

Due February 6

1. The language of sets consists of a single binary relation $\{\epsilon\}$. An *interpretation* (or a *model*) of the language of sets consists of a pair (D, E), where D is the *domain of discourse* and E is a binary relation on D which is the interpretation of the membership relation ϵ. Once we provide an interpretation of the language of sets, all sentences in the language become true or false.

The simplest way of describing the relation E (for finite models) is as a *directed graph* (or a *digraph.*) The domain of discourse D is the nodes of the graph, and the edges of the graph (represented by arrows) determine the membership relation E. If x and y are nodes of the graph then $x \in y$ is true if there is an arrow from x to y; and, $x \notin y$ if there is no arrow from x to y. For example, in the model **1e** below we have only $a \in b$ and $a \in c$ (so, $a \notin a$, $b \notin b$, $b \notin c$, $b \notin a$, $c \notin a$, $c \notin b$, $c \notin c$.) In this model the Axiom of Foundation is true:

- It holds for a because a has no elements (so the antecedent of the Axiom fails.)
- It holds for b because only $a \in b$, but a and b share no members (so the consequent of the Axiom is true.)
- It holds for c because only $a \in c$, but a and c share no members (so the consequent of the Axiom is true.)

On the other hand, the Axiom of Extensionality is false. I leave it to you to show why.

For each of the following interpretations of set theory state which of axioms 1,2,4,5 are true. Explain your answer.

1a. $D = \{a\}$, E is 

Proof. Note that $a = \emptyset$.

true. Axiom 1, Axiom 2, Axiom 5 ($\bigcup \emptyset = \emptyset$).

false.

- Axiom 4: there is no x with $a \to x$.

□

1b. $D = \{a\}$, E is

Proof. Note that $\emptyset$ does not exist in this model, so Axiom 3 (Comprehension) fails. We also have $\{a\} = a$.

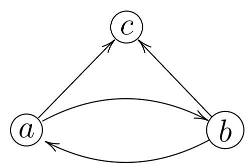
true. Axiom 1, Axiom 4 ($\{a, a\} = \{a\} = a$), Axiom 5 ($\bigcup a = a$).

false. Axiom 2: The following infinite backward chain is contrary to Foundation:

$$\ldots \to a \to a \to a.$$

□

1c. $D = \{a, b, c\}$, E is



Proof. Note that $\emptyset$ does not exist in this model, so Axiom 3 (Comprehension) fails.

true.

- Axiom 1
- Axiom 5:

$$\bigcup a = \{a\} = b \quad \bigcup b = \{b\} = a \quad \bigcup c = \{a, b\} = c.$$

false.

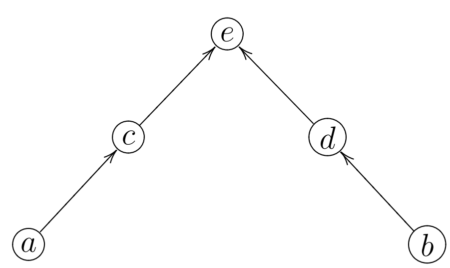
- Axiom 2: The following infinite backward chain is contrary to Foundation:

$$\ldots \to b \to a \to b \to a.$$

- Axiom 4: There is no set $\{c, c\} = \{c\}$.

□

1d. $D = \{a, b, c, d, e\}$, E is



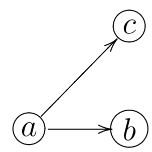
Proof. **true**. Axiom 2.

false.

- Axiom 1: There are two emptysets: a and b.
- Axiom 4: There is no set $\{e, e\} = \{e\}$ or $\{a, b\}$.
- Axiom 5: There is no set $\bigcup e = \{a, b\}$.

□

1e. $D = \{a, b, c\}$, E is



Proof. **true**.

- Axiom 2
- Axiom 5:

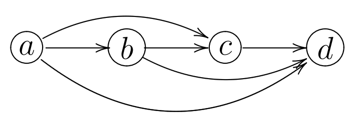
$$\bigcup a = \{\} = a \quad \bigcup b = \{\} = a \quad \bigcup c = \{\} = a.$$

false.

- Axiom 1: $b = \{a\}$ and $c = \{a\}$.
- Axiom 4: There is no set $\{c, c\} = \{c\}$ or $\{b, b\} = \{b\}$.

□

1f. $D = \{a, b, c, d\}$, E is



Proof. **true**.

- Axiom 1,2
- Axiom 5:

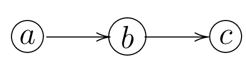
$$\bigcup a = \{\} = a \quad \bigcup b = \{\} = a \quad \bigcup c = \{a\} = b \quad \bigcup d = \{a, b\} = c$$

false.

- Axiom 4: There is no set $\{d, d\} = \{c\}$.

□

1g. $D = \{a, b, c\}$, E is



Proof. **true**.

- Axiom 1,2
- Axiom 5:

$$\bigcup a = \{\} = a \quad \bigcup b = \{\} = a \quad \bigcup c = \{a\} = b$$

false.

- Axiom 4: There is no set $\{d, d\} = \{c\}$.

□

2. Which of the models in **1** satisfies Axiom of Extensionality and the statement that there is no empty set ($\neg\exists x\forall y(y \notin x)$.)

Proof. (1b) and (1c) satisfy Extensionality, but there is no emptyset. □

3. Show that model **1a** satisfies the Axiom of Comprehension.

Proof. The only subset of $a = \{\}$ is $\{\}$, which is a itself. Since Comprehension always holds in a model which includes every subset of any set in the model, it holds in this model. □

4. Which of the models in **1** satisfies the Axioms of Extensionality and Comprehension but does not have pairwise unions – that is, the model will contain elements z, u but will contain no w satisfying $\forall x(x \in w \quad \leftrightarrow x \in z \vee x \in u)$.

Proof. Model **1g** satisfies Comprehension, since every subset of every set is in the model:

- The subsets of a are $\{\} = a$,
- The subsets of b are $\{\} = a$ and $\{a\} = b$,
- The subsets of c are $\{\} = a$ and $\{b\} = c$

It even satisfies unions, but not pairwise unions:

$$b \cup c = \{a, b\}.$$

1d does not satisfy pairwise unions either:

$$c \cup d = \{a, b\},$$

However, it does not satisfy Comprehension. Any finite model for comprehension must contain every subset of any set. The model also fails Extensionality. □

The following was originally a homework problem, that I decided to cut-out. I have included it here for your interest.

5. When you start adding axioms, it becomes harder to produce models. Prove that there is no finite model for the Axioms of Comprehension and Pairing. *Hint.* You only need the following two consequences of these axioms: $\exists x\forall y(y \notin x)$ (existence of $\emptyset$) and $\forall z\exists x\forall y(y \in x \leftrightarrow y = z)$ (existence of $\{z\}$). If you try to draw the membership digraph, you'll see why it can't be finite.

Proof. Define the following sets recursively,

$$T_0 = \emptyset$$
$$T_{n+1} = \{T_n\}.$$

Let $\mathcal{M}$ be any model satisfying the existence of the emptyset, and closed under singletons (that is, $\{x\} \in \mathcal{M}$ whenever $x \in \mathcal{M}$).

First, each $T_n \in \mathcal{M}$. This is by induction on n. $T_0 = \emptyset \in \mathcal{M}$. Given $T_n \in \mathcal{M}$, then $T_{n+1} = \{T_n\} \in \mathcal{M}$. So, for each n, $T_n \in \mathcal{M}$.

We need to show that the sets are distinct: if $n \neq m$ then $T_n \neq T_m$. Clearly, $T_0 \neq T_1$. Suppose $T_0, \ldots, T_n$ are each distinct. Then, $T_{n+1} = \{T_n\}$, so T_{n+1} is distinct from each of $T_0, \ldots, T_n$ since $T_n \in T_{n+1}$, but $T_n \neq T_k$ for any $k < n$, and T_k is the unique member of T_{k+1}. Thus, $T_0, \ldots, T_{n+1}$ are distinct.

□