MATH 582 HOMEWORK 2
WEEK 3

Winter, 2009

Due January 23

Exercise 1. *For all sets A and B,*

$$(a) \qquad A \approx B \;\rightarrow\; \mathcal{P}(A) \approx \mathcal{P}(B).$$

$$(b) \qquad A \approx B \;\rightarrow\; A^2 \approx B^2$$

The following definition is for Exercise 2:

Definition. A real number α is algebraic if it is the root of some polynomial

$$P(x) = a_0 + a_1 x + \ldots a_n x^n$$

with integer coefficients $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ (where $n \geq 1$ and $a_n \neq 0$). That is,

$$P(\alpha) = 0.$$

A number which is not algebraic is transcendental.

Liouville (1844) was the first to show the existence of transcendental numbers, using continued fractions. The simplest example is

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots$$

Charles Hermite first proved e was transcendental in 1873; and Ferdinand von Lindemann first proved π was transcendental in 1882.

Cantor gave the simplest and most powerful proof of the existence of transcendental numbers (1874), which showed that "almost every" real number is transcendental.

Exercise 2. *Show that the set of algebraic numbers is countable, and hence there exist transcendental numbers.*

Hint. First, show that the set of all polynomials with integer coefficients is countable. Then, use the fact that any polynomial has only finitely many (real) roots.

Exercise 3. *For every* $\alpha < \beta$ *where* α, β *are reals,* ∞ *or* $-\infty$*, construct bijections which prove the equinumerosities:*

$$(\alpha, \beta) \approx (0, 1) \approx \mathbb{R}.$$

Where

$$(\alpha, \beta) = \{x \in \mathbb{R} \mid \alpha < x < \beta\}.$$

Exercise 4. *For every* $\alpha < \beta$ *where* α, β *are reals, construct bijections which prove the equinumerosities:*

$$[\alpha, \beta) \approx [\alpha, \beta] \approx \mathbb{R}.$$

Where

$$\begin{aligned} [\alpha, \beta) &= \{x \in \mathbb{R} \mid \alpha \leq x < \beta\}, \\ [\alpha, \beta] &= \{x \in \mathbb{R} \mid \alpha \leq x \leq \beta\}. \end{aligned}$$

The next exercise was seen as highly counterintuitive when Cantor first proved it.

Exercise 5. *Show that* $\mathbb{R} \approx \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$*. Conclude that* $\mathbb{R} \approx \mathbb{R}^n$ *for every* $n \geq 2$*.*

Hint. You might find it easier to show that $\Delta \approx \Delta^2$*, then use Exercise 1a. If you feel truly stuck, see H+J exercise 5.2.7, for one approach.*