MATH 582 HOMEWORK 2
WEEK 3

Winter, 2009

Due January 23

Exercise 1. *For all sets A and B,*

$$(a) \qquad A \approx B \ \rightarrow \ \mathcal{P}(A) \approx \mathcal{P}(B).$$
$$(b) \qquad A \approx B \ \rightarrow \ A^2 \approx B^2$$

Proof. (a). Suppose $\pi : A \to B$ is a bijection which witnesses $A \approx B$. Then the map $\big(X \mapsto \pi[X]\big)$ defines a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

- The map is injective: If $X, Z \subseteq A$ and $X \neq Z$, then $\pi[X] \neq \pi[Z]$ since π is injective.
- The map is surjective: For each $Y \subseteq B$, let $X = \pi^{-1}[Y]$. Then, $\pi[X] = Y$, as is easily verified.

(b). Consider the bijection $\big((a_1, a_2) \mapsto (\pi(a_1), \pi(a_2))\big)$. □

The following definition is for Exercise 2:

Definition. A real number α is algebraic if it is the root of some polynomial

$$P(x) = a_0 + a_1 x + \ldots a_n x^n$$

with integer coefficients $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ (where $n \geq 1$ and $a_n \neq 0$). That is,

$$P(\alpha) = 0.$$

A number which is not algebraic is transcendental.

Liouville (1844) was the first to show the existence of transcendental numbers, using continued fractions. The simplest example is

$$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000\ldots$$

Charles Hermite first proved e was transcendental in 1873; and Ferdinand von Lindemann first proved π was transcendental in 1882.

Cantor gave the simplest and most powerful proof of the existence of transcendental numbers (1874), which showed that "almost every" real number is transcendental.

Exercise 2. *Show that the set of algebraic numbers is countable, and hence there exist transcendental numbers.*

Hint. First, show that the set of all polynomials with integer coefficients is countable. Then, use the fact that any polynomial has only finitely many (real) roots.

Proof. Let Π be the set of all polynomial with integer coefficients. Define a map from $p(x) \in \Pi$ by

$$\big(p(x) = a_0 + a_1 x + \dots a_n x^n \ \mapsto \ (a_0, a_1, \dots, a_n) \in \mathbb{Z}^{n+1}\big).$$

This is an injection from Π into the countable set $\bigcup_{n=2}^{\infty} \mathbb{Z}^n$. So, Π is countable. Fix an enumeration of Π

$$\Pi = \{p_n \,\big|\, n \in \mathbb{N}\}.$$

For each polynomial $p(x)$, the set of its roots

$$\Lambda(p) = \{\alpha \in \mathbb{R} \,\big|\, p(\alpha) = 0\},$$

is finite. The set of algebraic numbers is given by

$$\bigcup_{n \in \mathbb{N}} \Lambda(p_n),$$

which is a countable union of countable sets – so countable. □

Exercise 3. *For every $\alpha < \beta$ where α, β are reals, ∞ or $-\infty$, construct bijections which prove the equinumerosities:*

$$(\alpha, \beta) \approx (0, 1) \approx \mathbb{R}.$$

Where

$$(\alpha, \beta) = \{x \in \mathbb{R} \,\big|\, \alpha < x < \beta\}.$$

Proof. Let α and β be any real numbers. Then the map

$$\big(x \mapsto \alpha + x\beta\big)$$

is a bijection from $(0, 1)$ to (α, β).

We get a bijection from $(0, 1)$ to $(0, \infty)$ by the map

$$\big(x \mapsto -1 + \frac{1}{x}\big).$$

A bijection from $\mathbb{R} = (-\infty, \infty)$ to (α, ∞) (where α is any real number) is given by the map

$$\big(x \mapsto \alpha + e^x\big).$$

A bijection from from $\mathbb{R}$ to $(-\infty, \beta)$ (where β is any real number) is given by the map

$$\big(x \mapsto \beta - e^x\big).$$

So, putting these bijections together, we have for any α, β real, ∞ or $-\infty$,

$$(\alpha, \beta) \approx (0,1) \approx (-\infty, \infty) = \mathbb{R}.$$

□

Exercise 4. *For every $\alpha < \beta$ where α, β are reals, construct bijections which prove the equinumerosities:*

$$[\alpha, \beta) \approx [\alpha, \beta] \approx \mathbb{R}.$$

Where

$$\begin{aligned} [\alpha, \beta) &= \{x \in \mathbb{R} \,\big|\, \alpha \le x < \beta\}, \\ [\alpha, \beta] &= \{x \in \mathbb{R} \,\big|\, \alpha \le x \le \beta\}. \end{aligned}$$

Proof. The same maps in Exercise 3 extend to the intervals here, to show that

$$[0,1) \approx [\alpha, \beta) \qquad [0,1] \approx [\alpha, \beta]$$

when α and β are real numbers.

It is sufficient to define define bijections which show

$$(0,1) \approx [0,1) \approx [0,1]$$

It then follows that

$$\begin{aligned} [0,1) \approx [0,\infty) &\qquad \text{by } 0 \mapsto 0 \text{ and } (0,1) \rightleftarrows (0,\infty) \\ [0,1] \approx [0,\infty) &\qquad \text{by } 1 \mapsto 0 \text{ and } [0,1) \rightleftarrows (0,\infty) \\ [0,1) \approx \mathbb{R} &\qquad \text{by } [0,1) \rightleftarrows (0,1) \rightleftarrows \mathbb{R} \\ [0,1] \approx \mathbb{R} &\qquad \text{by } [0,1] \rightleftarrows (0,1) \rightleftarrows \mathbb{R} \end{aligned}$$

We define a bijection $\pi : [0,1) \rightleftarrows (0,1)$ by:

$$\pi(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{1}{n+1} \quad \text{where } n > 0 \\ x & \text{otherwise.} \end{cases}$$

For example,

$$\frac{1}{2} \xrightarrow{\pi} 1 \qquad \frac{1}{3} \xrightarrow{\pi} \frac{1}{2} \qquad \frac{1}{4} \xrightarrow{\pi} \frac{1}{3} \qquad \ldots$$

It is now straightforward to verify that π is a bijection.

We define another bijection $\rho : [0,1] \rightleftarrows [0,1)$ by:

$$\rho(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \quad \text{where } n > 0 \\ x & \text{otherwise.} \end{cases}$$

For example,

$$1 \xrightarrow{\rho} \frac{1}{2} \qquad \frac{1}{2} \xrightarrow{\rho} \frac{1}{3} \qquad \frac{1}{3} \xrightarrow{\rho} \frac{1}{4} \qquad \cdots$$

It is now straightforward to verify that ρ is a bijection.

□

The next exercise was seen as highly counterintuitive when Cantor first proved it.

Exercise 5. *Show that* $\mathbb{R} \approx \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$*. Conclude that* $\mathbb{R} \approx \mathbb{R}^n$ *for every* $n \geq 2$*.*

Hint. You might find it easier to show that $\Delta \approx \Delta^2$*, then use Exercise 1a. If you feel truly stuck, see H+J exercise 5.2.7, for one approach.*

Proof. Since $\Delta \approx \mathbb{R}$, it follows that $\Delta^2 \approx \mathbb{R}^2$ by Exercise 1a. It is a little easier to organize the argument using Δ in place of $\mathbb{R}$.

If $\delta_1, \delta_2 \in \Delta$, let $f(\delta_1, \delta_2) \in \Delta$ be defined as follows:

$$f(\delta_1, \delta_2)\,(n) = \begin{cases} \delta_1(n) & \text{if } n \text{ is odd ,} \\ \delta_2(n) & \text{if } n \text{ is even .} \end{cases}$$

It is straightforward to verify that $f : \Delta^2 \rightleftarrows \Delta$ is indeed a bijection. (I am writing $f(\delta_1, \delta_2)$ for $f\big((\delta_1, \delta_2)\big)$.) □