MATH 582 HOMEWORK 1
WEEK 2

Winter, 2009
Due January 23

Exercise 1. *Show that the following identity need not hold when $f : A \to B$ is not injective.*

$$f[X \cap Y] = f[X] \cap f[Y] \qquad X, Y \subseteq A.$$

Proof. Let $A = B = \{0, 1\}$ and define $f : A \to B$ by $f(0) = f(1) = 0$. Let $X = \{0\}$ and $Y = \{1\}$. Then

$$f[X] \cap f[Y] = \{0\} \quad \text{but} \quad f[X \cap Y] = f[\emptyset] = \emptyset.$$

So, $f[X \cap Y] \neq f[X] \cap f[Y]$.

□

Exercise 2. *Prove the following identities. For every $f : A \to B$ and all sequences of sets $Y_n \subseteq B$ and $X_n \subseteq A$,*

$$(a) \qquad f^{-1}\Big[\bigcup_{n=0}^{\infty} Y_n\Big] = \bigcup_{n=0}^{\infty} f^{-1}[Y_n],$$

$$(b) \qquad f^{-1}\Big[\bigcap_{n=0}^{\infty} Y_n\Big] = \bigcap_{n=0}^{\infty} f^{-1}[Y_n],$$

$$(c) \qquad f\Big[\bigcup_{n=0}^{\infty} X_n\Big] = \bigcup_{n=0}^{\infty} f[X_n],$$

Proof. Let $Y_n \subseteq B$ and $X_n \subseteq A$ be sequences of sets and $f : A \to B$. We will use the Axiom of Function Identity (see Lecture 3, Slide 12).

(a). Let $x \in A$. Then

$$\begin{aligned} x \in f^{-1}\Big[\bigcup_{n=0}^{\infty} Y_n\Big] &\leftrightarrow f(x) \in \bigcup_{n=0}^{\infty} Y_n \\ &\leftrightarrow f(x) \in Y_n \quad \text{for some } n \\ &\leftrightarrow x \in f^{-1}[Y_n] \quad \text{for some } n \\ &\leftrightarrow x \in \bigcup_{n=0}^{\infty} f^{-1}Y_n]. \end{aligned}$$

(b). Let $x \in A$. Then

$$\begin{aligned} x \in f^{-1}\Big[\bigcap_{n=0}^{\infty} Y_n\Big] &\leftrightarrow f(x) \in \bigcap_{n=0}^{\infty} Y_n \\ &\leftrightarrow f(x) \in Y_n \quad \text{for every } n \\ &\leftrightarrow x \in f^{-1}[Y_n] \quad \text{for every } n \\ &\leftrightarrow x \in \bigcap_{n=0}^{\infty} f^{-1}[Y_n]. \end{aligned}$$

(c). Let $y \in B$. Then

$$\begin{aligned} y \in f\Big[\bigcup_{n=0}^{\infty} X_n\Big] &\leftrightarrow f(x) = y \quad \text{for some } x \in \bigcup_{n=0}^{\infty} X_n \\ &\leftrightarrow f(x) = y \quad \text{for some } x \in X_n \quad \text{and some } n \\ &\leftrightarrow y \in f[X_n] \quad \text{for some } n \\ &\leftrightarrow y \in \bigcup_{n=0}^{\infty} f[X_n]. \end{aligned}$$

□

Exercise 3. *Prove. For every injection $f : A \hookrightarrow B$ and every sequence of sets $X_n \subseteq A$,*

$$f\Big[\bigcap_{n=0}^{\infty} X_n\Big] = \bigcap_{n=0}^{\infty} f[X_n]$$

Proof. Let $f : A \hookrightarrow B$ and $X_n \subseteq A$ be a sequence of sets. We first argue without using injectivity.

$$\begin{aligned} y \in f\Big[\bigcap_{n=0}^{\infty} X_n\Big] &\leftrightarrow f(x) = y \quad \text{for some } x \in \bigcap_{n=0}^{\infty} X_n \\ &\leftrightarrow f(x) = y \quad \text{for some } x \text{ and every } n,\ x \in X_n \\ &\rightarrow y \in f[X_n] \quad \text{for every } n \\ &\leftrightarrow y \in \bigcap_{n=0}^{\infty} f[X_n]. \end{aligned}$$

We finish off the proof by reversing the third arrow. Suppose $y \in f[X_n]$ for every n. Then there is some $x \in A$ with $f(x) = y$, and in fact, this x is unique by the injectivity of f. So, $x \in X_n$ for every n.

□

Exercise 4. *Let A_n and B_n be sequences of sets. Prove the following identities.*

$$(a) \qquad \forall n \left[A_n \subseteq C\right] \rightarrow \bigcup_{n=0}^{\infty} A_n \subseteq C,$$

$$(b) \qquad \bigcup_{n=0}^{\infty}(A_n \cap C) = \Big(\bigcup_{n=0}^{\infty} A_n\Big) \cap C,$$

$$(c) \qquad \bigcup_{n=0}^{\infty}(A_n \cup B_n) = \bigcup_{n=0}^{\infty} A_n \cup \bigcup_{n=0}^{\infty} A_n.$$

Proof. Let A_n and B_n be sequences of sets. Note that $A_k \subseteq \bigcup_{n=0}^{\infty} A_n$ for every k. (And similarly for the B_n's.)

(a). Suppose for every n, $A_n \subseteq C$.

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty} A_n \quad &\rightarrow \quad x \in A_n \qquad \text{for some } n \\ &\rightarrow \quad x \in C. \end{aligned}$$

So, $\bigcup_{n=0}^{\infty} A_n \subseteq C$.

(b). Then,

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty}(A_n \cap C) \quad &\leftrightarrow \quad x \in A_n \cap C \qquad \text{for some } n \\ &\leftrightarrow \quad x \in A_n \;\wedge\; x \in C \qquad \text{for some } n \\ &\leftrightarrow \quad x \in \bigcup_{n=0}^{\infty} A_n \;\wedge\; x \in C \\ &\leftrightarrow \quad \Big(\bigcup_{n=0}^{\infty} A_n\Big) \cap C \end{aligned}$$

(c).

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty}(A_n \cup B_n) \quad &\leftrightarrow \quad x \in A_n \cup B_n \qquad \text{for some } n \\ &\leftrightarrow \quad x \in A_n \;\vee\; x \in B_n \qquad \text{for some } n \\ &\leftrightarrow \quad x \in \bigcup_{n=0}^{\infty} A_n \;\vee\; x \in \bigcup_{n=0}^{\infty} B_n \\ &\leftrightarrow \quad x \in \bigcup_{n=0}^{\infty} A_n \cup x \in \bigcup_{n=0}^{\infty} B_n \end{aligned}$$

$\square$