MATH 582 HOMEWORK
WEEK 14

Winter, 2009
Due April 20

Introduction

This is a full complement of problems. Let your time and interests dictate how much you choose to do. The most important problems (and most interesting I hope) are problems 6-9 on equivalents of (CH). Give these a try first. Problems 1-5 are more straightforward applications in cardinal arithmetic.

Cardinal arithmetic

Exercise 1. *Prove the following equations.*
(a) $\prod_{i\in I}\left(\kappa^{\lambda_i}\right) = \kappa^{\sum_{i\in I}\lambda_i}$.
(b) $\left(\prod_{i\in I}\kappa_i\right)^{\lambda} = \prod_{i\in I}\kappa_i^{\lambda}$.
Hint. Generalize the argument of Lecture 24, slide 15 or H+J Theorem 5.1.7, which is the case for I finite.

Exercise 2. *If* $\aleph_\alpha$ *is strongly inaccessible and* $\beta < \alpha$*, then* $\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha$*. (See Lecture 32, slide 12, or H+J page 168, for the definition of strongly inaccessible.)*

Exercise 3. *Let* κ *be an infinite cardinal. Then* κ *can be partitioned into a family of sets* $\{A_\eta \mid \eta < \kappa\}$ *which are each unbounded in* κ*, mutually disjoint and with* $\kappa = \cup_{\eta<\kappa}A_\eta$*.*
Hint. Think about the construction in the theorem that $\kappa\cdot\kappa = \kappa$ *from Lecture 27.*

Exercise 4. *Let* κ *be an infinite cardinal and suppose* $\langle \kappa_\xi \mid \xi < cf(\kappa)\rangle$ *is a strictly increasing cofinal sequence of cardinals with* $\kappa = \sup_{\xi<cf(\kappa)}\kappa_\xi$ *and* $\kappa_0 > 0$*. Show that*

$$\kappa^{cf(\kappa)} = \prod_{\xi<cf(\kappa)}\kappa_\xi.$$

Hint. From the previous problem there is a family of sets $\{A_\eta \mid \eta < cf(\kappa)\}$ *which partitions* $cf(\kappa)$*. Then show*

$$\kappa^{cf(\kappa)} \le \prod_{\eta<cf(\kappa)}\Big(\sum_{\xi\in A_\eta}\kappa_\xi\Big) \le \prod_{\eta<cf(\kappa)}\Big(\prod_{\xi\in I_\eta}\kappa_\xi\Big) \le \prod_{\eta<cf(\kappa)}\kappa_\xi \le \kappa^{cf(\kappa)}$$

Exercise 5. *If* $2^{\aleph_1} = \aleph_2$*, then* $\aleph_\omega^{\aleph_0} \neq \aleph_{\omega_1}$*.*
Hint. Use Bernstein-Hausdorff-Tarski to show $\aleph_n^{\aleph_1} = \aleph_n$ *for* $n \geq 2$*. Then conclude* $\aleph_\omega^{\aleph_0} = \aleph_\omega^{\aleph_1}$*. Alternatively, you can conclude this from exercise 9.3.7 in H+J whose proof is analogous to Bernstein's Theorem of Lecture 33, slide 10.*

Equivalents to the Continuum Hypothesis

Exercise 6. *Prove that the Continuum Hypothesis (CH) is equivalent to the statement:*

> $\mathbb{R}^2 = A \cup B$ *where* A *intersects every horizontal line and* B *intersects every vertical line in countably many points.*

This decomposition of the plane is called the Sierpinski Decomposition.
Hint. ($\Rightarrow$)*: (CH) implies* $|\mathbb{R}| = \aleph_1$*, so write* $\mathbb{R}$ *as* $\{r_\xi \,|\, \xi < \omega_1\}$*. Let* $(r_\xi, r_\eta) \in A$ *iff* $\xi < \eta$*.* ($\Rightarrow$)*: Assume (CH) fails, so that* $|\mathbb{R}| \geq \omega_2$*, but that there exists a Sierpinski decomposition. Fix a set* $U \subset \mathbb{R}$ *with* $|U| = \omega_1$*. Show that for every* $r \in \mathbb{R}$ *there is a* $u \in U$ *with* $(r, u) \in A$*; then, show that there is a* $u \in U$ *with* $\{(r, u) \in A \,|\, r \in \mathbb{R}\}$ *uncountable.*

Exercise 7. *Chris Freiing proposed the following as an "axiom" (a statement which ought to be taken as true):*

> *For every function* F *mapping reals in the unit interval* $[0, 1]$ *to countable subsets of* $[0, 1]$*, there exists an* x *and* y *such that* $x \notin F(y)$ *and* $y \notin F(x)$*.*

Show that Freiling's axiom is equivalent to the negation of (CH). Hint. Show the contrapositive. The argument is similar to that in Problem 6. This equivalence is actually due to Sierpinski; Freiling uses the intuitive plausibility of the statement to argue against the Continuum Hypothesis.

Exercise 8. *Prove that the Continuum Hypothesis (CH) is equivalent to the statement:*

> *There is a decomposition of* $\mathbb{R}^3 = A_1 \cup A_2 \cup A_3$ *such that if* L *is a line parallel to the* x_i*-axis then* $L \cap A_i$ *is finite.*

Hint. ($\Rightarrow$)*: Fix* $\phi_\gamma : \gamma + 1 \xrightarrow{1\text{-}1} \omega$*. Let* $\mathbb{R} = \{r_\xi \,|\, \xi < \omega_1\}$*. For any triple* (x_1, x_2, x_3) *where the entries are* $r_\alpha, r_\beta, r_\gamma$ *(in some order), if* $\alpha, \beta \leq \gamma$ *and* $\phi_\gamma(\alpha) \leq \phi_\gamma(\beta)$ *then put* $(x_1, x_2, x_3) \in A_i$ *where* $x_i = r_\alpha$*.* ($\Leftarrow$)*: Suppose* $|\mathbb{R}| \geq \aleph_2$*, but there is a decomposition as in the statement. Fix* $U, V, W \subseteq \mathbb{R}$ *with* $|U| = \aleph_0$*,* $|V| = \aleph_1$ *and* $|W| = \aleph_2$*. Show that there is* *a* $c \in W$ *such that there are no* $u \in U, v \in V$ *with* $(u, v, c) \in A_3$. *Fix* c, *and show that there is a* $b \in V$ *such that for no* $u \in U$ *is* $(u, b, c) \in A_2$. *Fix such a* b, *and show that there is an* $a \in U$ *with* $(a, b, c) \notin A_1$.

Exercise 9. *Here is an alternative to Freiling's Axiom in Problem 7 which seems intuitively plausible:*

> *For every function* F *mapping unordered pairs of reals in the unit interval* $[0, 1]$ *to finite subsets of* $[0, 1]$, *there exists reals* x, y *and* z *such that* $x \notin F(\{y, z\})$ *and* $y \notin F(\{x, z\})$ *and* $z \notin F(\{x, y\})$.

Show that this statement is equivalent to the negation of (CH). Hint. Show the contrapositive. The argument is similar to that in Problem 8.