MATH 582 HOMEWORK 7
WEEK 12

Winter, 2009

Due April 20

1. Axiom of Choice and Equivalents

You may use the Axiom of Choice (AC) or one of its equivalents from Lecture 28.

Exercise 1. *Prove the following statement is equivalent to the Axiom of Choice:*

> *Any surjective function has a right inverse. (That is, if $f : A \twoheadrightarrow B$ then there is a function $g : B \to A$ such that $f \circ g(x) = x$ for every $x \in B$.)*

Note that when A is well-ordered the statement can be proven without using AC (see Exercise 7.2.6 in Hrbacek + Jech.)

Exercise 2. *(AC) Every partial ordering can be extended to a linear ordering. That is, if $(A, \lhd)$ is a partial ordering then there is a linear ordering $<$ on A with $\lhd \subseteq <$. (The order $<$ is sometimes called the linearization of $\lhd$.)*

2. Weaker Versions of the Axiom of Choice

Let $\mathcal{F}$ be a family of nonempty sets. A *choice function* for $\mathcal{F}$ is a function $g : \mathcal{F} \to \bigcup \mathcal{F}$ satisfying $g(X) \in X$ for all $X \in \mathcal{F}$.

> **Axiom of Countable Choice**: Every countable family of nonempty sets has a choice function.

Exercise 3. *Let $A \subseteq \mathbb{R}$. An accumulation point for A is a real number r such that $\forall \epsilon > 0 \, \exists a \in A \left(|a - r| < \epsilon \right)$.*

(a) Prove, using the Axiom of Countable Choice, that if r is an accumulation point for a set of reals A, then there is a countable sequence $\langle a_n \mid n < \omega \rangle$ of members of A with $\lim_{n\to\infty} a_n = r$.

(b) Prove (a) when A is a set of rational numbers without using the Axiom of Countable Choice.

> **Axiom of Dependent Choices**: Let A be a nonempty set and R a relation on A satisfying the condition that for every x there is a y with xRy. Then there is a sequence $\langle a_n \mid n < \omega \rangle$ of elements of A such that $a_n R a_{n+1}$ for every $n < \omega$.

Exercise 4. *(AC) Show the Axiom of Dependent Choices follows from the Axiom of Choice.*

Exercise 5. *Show the Axiom of Countable Choice follows from the Axiom of Dependent Choices.*

The Axiom of Dependent Choices is actually stronger then the Axiom of Countable Choice. To see why this might be so: the Axiom of Countable Choice says that if we are given a countable family of nonempty sets $\langle X_n \mid n < \omega \rangle$ *ahead of time* then we have a choice function for this family. On the other hand with the Axiom of Dependent Choices we choose a_n from the set $\{x \mid a_{n-1} R x\}$, which we can only know once we have chosen a_{n-1}; so the sets for which we are constructing a choice function are only given to us *during the course of construction*. This is reflected in your proof in Problem **6**.

Exercise 6. *A relation $R \subseteq A \times A$ is well-founded if every nonempty $X \subseteq A$ has an R-minimal element (that is, there is a $z \in X$ such that $\neg(yRz)$ for every $y \in X$.) Prove the Axiom of Dependent Choice is equivalent to the following statement:*

> *If a relation $R \subseteq A \times A$ is not well-founded then there is a sequence $\langle a_n \mid n < \omega \rangle$ such that $a_{n+1} R a_n$ for every $n < \omega$. (Such a sequence is called a descending R chain.)*