MATH 582 HOMEWORK 6
WEEK 10

Winter, 2009
Due April 10

Exercise 1. *Counting arguments.*

(a) *A sequence $\langle a_n \,|\, n < \omega \rangle$ of natural numbers is eventually constant if there is a $k \in \omega$ such that $s_k = s_n$ for all $n \geq k$. Show that the set of eventually constant sequences of natural numbers is countable.*

(b) *A sequence $\langle a_n \,|\, n < \omega \rangle$ of natural numbers is periodic if there are $k, p \in \omega$ with $p > 0$ such that for every $n \geq k$, $s_n = s_{n+p}$. Show that the set of periodic sequences of natural numbers is countable.*

The next 5 problems (**2**-**6**) will reconstruct Zermelo's proof of the Schröder-Bernstein theorem. Zermelo based his proof on ideas of Dedekind using fixed-points of monotone functions (which generalize features of normal functions on the ordinals).

Definition. Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$. F is a *monotone* function if $X \subseteq Y \subseteq A$ implies $F(X) \subseteq F(Y)$. A set $X \subseteq A$ is a *fixed point* of F if $F(X) = X$.

Exercise 2. *Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$ be a monotone map.*

(a) *Show that F has a fixed point C.*
Hint. Let $\mathcal{C} = \{X \subseteq A \,|\, F(X) \subseteq X\}$. Show $\mathcal{C} \neq \emptyset$, and let $C = \bigcap \mathcal{C}$; now show that both $C, F(C) \in \mathcal{C}$ and conclude C is a fixed point of F.

(b) *Show that the fixed point C constructed in (a) is the least fixed point: if $D \subseteq A$ is any other fixed point of F then $C \subseteq D$.*

Exercise 3. *Prove: If $A' \subseteq B \subseteq A$ and $A' \approx A$ then $A \approx B$.*
Hint. Let $f : A \rightleftarrows A'$; define $G : \mathcal{P}(A) \to \mathcal{P}(A)$ by $G(X) = A - B \cup f[X]$. Show G has a fixed point C. Now, define $g : A \rightleftarrows B$ by

$$g(x) = \begin{cases} ?? & \text{if } x \in C \\ ?? & \text{if } x \in A - C \end{cases}$$

Exercise 4. *Use **3** to give a proof of the Schröder-Bernstein Theorem: if* $A \preccurlyeq B$ *and* $B \preccurlyeq A$ *then* $A \approx B$.

Definition. A sequence of sets $\langle A_n \mid n < \omega \rangle$ is *nondecreasing* if $A_n \subseteq A_{n+1}$. A function $F : \mathcal{P}(A) \to \mathcal{P}(A)$ is *continuous* if $F\big(\bigcup_{n\in\omega} A_n\big) = \bigcup_{n\in\omega} F(A_n)$ for every nondecreasing sequence of sets.

Exercise 5. *Prove that the function* G *from **3** is continuous.*

Exercise 6. *Let* $F : \mathcal{P}(A) \to \mathcal{P}(A)$ *be a continuous and monotone function. Define a sequence of sets* $\langle A_n \mid n < \omega \rangle$ *by primitive recursion as follows:* $A_0 = \emptyset$*; given* A_n *define* $A_{n+1} = F(A_n)$*. Prove that* $C = \bigcup_{n<\omega} A_n$ *is the least fixed point of* F.

*Note. Compare the construction here to that of **2**: in the earlier exercise we constructed the fixed point "from above"; here with the stronger condition of continuity, we can construct a fixed point "from below". The two constructions produce the same fixed point. Now, if you think back to our original construction of the Schröder-Bernstein Theorem, we were really constructing a fixed-point "from below" for the function*

$$X \mapsto (A - g(B)) \cup g \circ f(X),$$

where the functions f, g *are the 1-1 maps given by* $A \preccurlyeq B$ *and* $B \preccurlyeq A$.

Monotone continuous set functions are a generalization of the normal functions *on the ordinals. Verify that normal functions are really monotone and continuous.*

The next 4 problems (**7-10**) introduce an alternative definition of finite/infinite set. A set X is *Dedekind infinite* if there is a 1-1 mapping of X onto a proper subset. A set is *Dedekind finite* if it is not Dedekind infinite.

Exercise 7. *If* X *contains a countably infinite subset, then* X *is Dedekind infinite.*

Exercise 8. *If* X *is Dedekind infinite, then it contains a countably infinite subset. Hint. This is a modification of the idea in the Schröder-Bernstein Theorem: Let* f *be the embedding of* X *into a proper subset*

$f[X]$. Choose any $x \in X - f[X]$. Consider the set $\{f^n(x) \,\big|\, n < \omega\}$, where $f^n(x)$ is the result of n applications of f to x: $x, f(x), f(f(x)), f(f(f(x))), \ldots$.

Note. It follows from **7** and **8** that the Dedekind infinite sets are exactly the infinite sets with a countably infinite subset. We cannot yet prove that *every infinite set* has a countably infinite subset (see Discussion in Lecture 25, slide 18.) Once we have the Axiom of Choice, we will be able to prove that Infinite=Dedekind Infinite.

Exercise 9. *If A and B are Dedekind finite, then so is $A \cup B$ and $A \times B$.*

Exercise 10. *If A is infinite, then $\mathcal{P}(\mathcal{P}(A))$ is Dedekind infinite.*
Hint. Consider the sets $S_n = \{X \subset A \,\big|\, |X| = n\}$.

Exercise 11. *Use the Axiom of Choice (which is required for this problem).*
Prove that the Dedekind finite sets are finite.